Rel. kv. mek   8.6.90

Løsninger

1 a) Kanonisk form $H\psi = [c\,\vec{\alpha}(\vec{p} - e\vec{A}) + \beta mc^2 + e\varphi]\psi = E\psi$ Elektronlad. $e = -|e|$

Med $\vec{p} = -i\hbar\nabla$, $E = i\hbar\frac{\partial}{\partial t}$ (operatorer)

$$(c\vec{\alpha}(-i\hbar\nabla - e\vec{A}) + \beta mc^2 + e\varphi)\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Her oppfyller Dirac-matrisene:

$$[\alpha_i, \alpha_k]_+ = \alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik} \qquad i,k = 1,2,3$$

$$[\beta, \alpha_i]_+ = 0 \qquad \beta^2 = 1$$

Kovariant form.

$$(c\,\gamma^\mu(p_\mu - eA_\mu) - mc^2) = 0 \qquad \text{med } [\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad \mu,\nu = 0,1,2,3$$

Metrisk tensor $g^{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu \neq 0 \\ 0 & \mu \neq \nu \end{cases}$ $\gamma^0 = \beta$, $\vec{\gamma} = \beta\vec{\alpha}$

$$(c\gamma^\mu(i\hbar\partial_\mu - eA_\mu) - mc^2)\psi = 0 \qquad p_\mu = i\hbar\frac{\partial}{\partial x^\mu},\ A_\mu = \left(\frac{\varphi}{c}, -\vec{A}\right)$$

b) Ved en Lorentz-transformasjon transformerer vektorer slik

$x^{\mu\prime} = a^\mu{}_\nu x^\nu$ med $a^\mu{}_\nu a_\mu{}^\lambda = \delta_\nu{}^\lambda$ (indeksen refererer til romkoordinatene)

mens en Dirac-spinor transformerer slik

$\psi_\sigma' = S_{\sigma\beta}\psi_\beta$ (indeksen refererer til spinorkomponentene)

Sammenhengen mellom Dirac-ligningen i de to inertialsystemene

$$(c\,\gamma^\nu(p_\nu' - eA_\nu') - mc^2)\psi' = (c\gamma^\nu a_\nu{}^\mu(p_\mu - eA_\mu) - mc^2)S\psi = 0$$

Multipliserer med S^{-1} fra venstre

$$(S^{-1}\gamma^\nu S\, a_\nu{}^\mu(p_\mu - eA_\mu) - mc^2)\psi = 0$$

Invariant hvis dette er lik

$$(c\gamma^\mu(p_\mu - eA_\mu) - mc^2)\psi = 0$$

d.v.s betingelse

$$S^{-1}\gamma^\nu S\, a_\nu{}^\mu = \gamma^\mu$$

eller multiplisert med $a^\lambda{}_\mu$

$$S^{-1}\gamma^\nu S\,\delta_\nu^\lambda = a^\lambda{}_\mu\gamma^\mu \quad\Rightarrow\quad \underline{S^{-1}\gamma^\lambda S = a^\lambda{}_\mu\gamma^\mu}$$

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1c) Når bevegelse langs x-aksen hvor

$$a^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{med} \quad \cosh\alpha = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} = \frac{E}{mc^2}, \quad \sinh\alpha = \frac{v/c}{\sqrt{1-(\frac{v}{c})^2}} = \frac{p}{mc}$$

vil betingelsen være oppfyllt med

$$S = 1\cosh\frac{\alpha}{2} + \gamma^0\gamma^1 \sinh\frac{\alpha}{2}$$

Har $S^{-1} = 1\cosh\frac{\alpha}{2} - \gamma^0\gamma^1\sinh\frac{\alpha}{2}$ Vises ved utregning, eller finnes f.eks. fra

$$S^{-1}S = (A + \gamma^0\gamma^1 B)(\cosh\tfrac{\alpha}{2} + \gamma^0\gamma^1\sinh\tfrac{\alpha}{2}) =$$
$$= (A\cosh\tfrac{\alpha}{2} + B\sinh\tfrac{\alpha}{2}) + (A\sinh\tfrac{\alpha}{2} + B\cosh\tfrac{\alpha}{2})\gamma^0\gamma^1$$
$$= 1 \quad \text{som gir} \quad A = \cosh\tfrac{\alpha}{2},\ B = -\sinh\tfrac{\alpha}{2}.$$

Innsatt i betingelsesligningen

$$(\cosh\tfrac{\alpha}{2} - \gamma^0\gamma^1\sinh\tfrac{\alpha}{2})\gamma^\lambda(\cosh\tfrac{\alpha}{2} + \gamma^0\gamma^1\sinh\tfrac{\alpha}{2}) = a^\lambda{}_0\gamma^0 + a^\lambda{}_1\gamma^1 + a^\lambda{}_2\gamma^2 + a^\lambda{}_3\gamma^3$$

$\lambda = 0$

$$\cosh^2\tfrac{\alpha}{2}\,\gamma^0 - \gamma^0\gamma^1\underbrace{\gamma^0\gamma^0}_{1}\gamma^1\sinh^2\tfrac{\alpha}{2} + \cosh\tfrac{\alpha}{2}\sinh\tfrac{\alpha}{2}\underbrace{\gamma^0\gamma^0}_{1}\gamma^1 - \sinh\tfrac{\alpha}{2}\cosh\tfrac{\alpha}{2}\underbrace{\gamma^0\gamma^1\gamma^0}_{-\gamma^0\gamma^1}$$
$$= (\cosh^2\tfrac{\alpha}{2} + \sinh^2\tfrac{\alpha}{2})\gamma^0 + 2\cosh\tfrac{\alpha}{2}\sinh\tfrac{\alpha}{2}\,\gamma^1 = \underline{\cosh\alpha\,\gamma^0 + \sinh\alpha\,\gamma^1} \quad \text{O.K}$$

$\lambda = 1$

$$\cosh^2\tfrac{\alpha}{2}\,\gamma^1 - \underbrace{\gamma^0\gamma^1\gamma^1}_{-1}\gamma^0\gamma^1\sinh^2\tfrac{\alpha}{2} + \cosh\tfrac{\alpha}{2}\sinh\tfrac{\alpha}{2}\underbrace{\gamma^1\gamma^0\gamma^1}_{-\gamma^0\gamma^1} - \sinh\tfrac{\alpha}{2}\cosh\tfrac{\alpha}{2}\,\gamma^0\underbrace{\gamma^1\gamma^1}_{-1}$$
$$= (\cosh^2\tfrac{\alpha}{2} + \sinh^2\tfrac{\alpha}{2})\gamma^1 + 2\cosh\tfrac{\alpha}{2}\sinh\tfrac{\alpha}{2}\,\gamma^0 = \underline{\cosh\alpha\,\gamma^1 + \sinh\alpha\,\gamma^0} \quad \text{O.K}$$

$\lambda = 2, 3$

$$\cosh^2\tfrac{\alpha}{2}\,\gamma^{2,3} - \gamma^0\gamma^1\underbrace{\gamma^{2,3}\gamma^0\gamma^1}_{\gamma^0\gamma^1\gamma^{2,3}}\sinh^2\tfrac{\alpha}{2} + \cosh\tfrac{\alpha}{2}\sinh\tfrac{\alpha}{2}\,\gamma^{2,3}\gamma^0\gamma^1 - \sinh\tfrac{\alpha}{2}\cosh\tfrac{\alpha}{2}\underbrace{\gamma^0\gamma^1\gamma^{2,3}}_{\gamma^{2,3}\gamma^0\gamma^1}$$
$$= (\cosh^2\tfrac{\alpha}{2} - \sinh^2\tfrac{\alpha}{2})\gamma^{2,3} = \underline{\gamma^{2,3}} \quad \text{O.K}$$

1d) Dirac-ligning for fritt elektron i ro: $\vec{p} = 0 \quad p^0 = \frac{E}{c} = mc$

$$(\gamma^\mu p_\mu - mc)\psi = 0 \qquad i\hbar\nabla\psi = \vec{p}\psi = 0 \qquad i\hbar\frac{\partial}{\partial t}\psi = E_0\psi = mc^2\psi$$

$$(\gamma^0 i\hbar\tfrac{\partial}{\partial t} - mc^2)\psi = 0$$

I standardrepresentasjonen

$$\left[i\hbar\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}\frac{\partial}{\partial t} - \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}mc^2\right]\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix} = 0 \Rightarrow \begin{aligned} i\hbar\tfrac{\partial}{\partial t}\psi_{1,2} - mc^2\psi_{1,2} &= 0 \\ -i\hbar\tfrac{\partial}{\partial t}\psi_{3,4} - mc^2\psi_{3,4} &= 0\end{aligned}$$

$$\Rightarrow \begin{aligned}\psi_{1,2} &= u_{1,2}\,e^{-\frac{i}{\hbar}E_0 t} \\ \psi_{3,4} &= u_{3,4}\,e^{\frac{i}{\hbar}E_0 t}\end{aligned}$$

Positiv-energi-løsning: $\underline{\psi_0 = \left(\frac{1}{2\pi\hbar}\right)^{3/2}\begin{pmatrix}u_1\\u_2\\0\\0\end{pmatrix}e^{-\frac{i}{\hbar}E_0 t}}$ $\qquad E_0 = mc^2$

normert til $\int\psi_0^+\psi\,d^3r = 1$ når $|u_1|^2 + |u_2|^2 = 1$.

1e) Sammenheng mellom ψ i to inertialsystem $\psi' = S\psi$.

Fasen er invariant $p'_\mu x^{\mu'} = a_\mu{}^\lambda a^\mu{}_\nu p_\lambda x^\nu = \delta^\lambda{}_\nu p_\lambda x^\nu = p_\nu x^\nu$

Velger x-aksen langs L''s bevegelse:

$$Et - \vec{p}\vec{r} = E_0 t_0$$

$$S = \cosh\frac{\alpha}{2}\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \sinh\frac{\alpha}{2}\begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix} = \begin{pmatrix}\cosh\frac{\alpha}{2}&&&\sinh\frac{\alpha}{2}\\&\cosh\frac{\alpha}{2}&\sinh\frac{\alpha}{2}&\\&\sinh\frac{\alpha}{2}&\cosh\frac{\alpha}{2}&\\\sinh\frac{\alpha}{2}&&&\cosh\frac{\alpha}{2}\end{pmatrix}$$

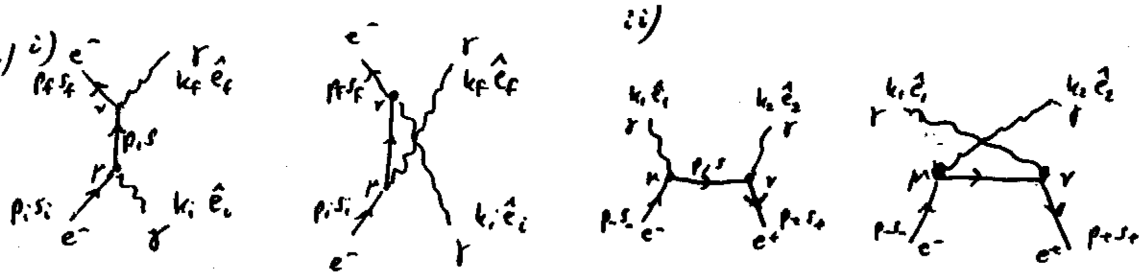
da $\gamma^0\gamma^1 = \beta\beta\alpha^1 = \alpha^1 = \begin{pmatrix}0&\sigma^1\\\sigma^1&0\end{pmatrix}$

$$\psi' = \left(\frac{1}{2\pi\hbar}\right)^{3/2} S\,u\,e^{\frac{i}{\hbar}(\vec{p}\vec{r}-Et)} = \left(\frac{1}{2\pi\hbar}\right)^{3/2}\begin{pmatrix}\cosh\frac{\alpha}{2}\,u_1\\\cosh\frac{\alpha}{2}\,u_2\\\sinh\frac{\alpha}{2}\,u_2\\\sinh\frac{\alpha}{2}\,u_1\end{pmatrix}e^{\frac{i}{\hbar}(\vec{p}\vec{r}-Et)}$$

$$= \left(\frac{1}{2\pi\hbar}\right)^{3/2}\sqrt{\frac{E+mc^2}{2mc^2}}\begin{pmatrix}u_1\\u_2\\\frac{cp_x}{E+mc^2}u_2\\\frac{cp_x}{E+mc^2}u_1\end{pmatrix}e^{\frac{i}{\hbar}(p_x x-Et)}$$

Har benyttet
$$\cosh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha+1}{2}} = \sqrt{\frac{E+mc^2}{2mc^2}}$$
$$\sinh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha-1}{2}} = \sqrt{\frac{E-mc^2}{2mc^2}} = \sqrt{\frac{E+mc^2}{2mc^2}}\frac{\sqrt{E^2-(mc^2)^2}}{E+mc^2} = \sqrt{\frac{E+mc^2}{2mc^2}}\frac{p_x c}{E+mc^2}$$

2 a) i)



b) i) Comptonspredning:

$$S^{(i)} = \left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{m}{E_f}\right)^{1/2}(-ie)(2\pi)^4\left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{1}{2\omega_f}\right)^{1/2}\frac{i}{(2\pi)^4}(-ie)(2\pi)^4\left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{1}{2\omega_i}\right)^{1/2}\left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{m}{E_i}\right)^{1/2}$$

$$\Big[\int d^4p\ \bar{u}_f\gamma^\nu\hat{e}_{f\nu}\frac{\gamma p+m}{p^2-m^2}\gamma^\mu\hat{e}_{i\mu}u_i\qquad\delta^4(p-p_f-k_f)\,\delta^4(p_i+k_i-p)$$

$$+\int d^4p\ \bar{u}_f\gamma^\nu\hat{e}_{i\nu}\frac{\gamma p+m}{p^2-m^2}\gamma^\mu\hat{e}_{f\mu}u_i\qquad\delta^4(p-p_f+k_i)\,\delta^4(p_i-p-k_f)\Big]$$

$$=\left(\frac{1}{2\pi}\right)^6\sqrt{\frac{m^2}{E_iE_f}}\ \frac{-ie^2}{2\sqrt{\omega_i\omega_f}}\left[\bar{u}_f\,\hat{\not e}_f\frac{\not p_i+\not k_i+m}{(p_i+k_i)^2-m^2}\hat{\not e}_i u_i+\bar{u}_f\,\hat{\not e}_i\frac{\not p_i-\not k_f+m}{(p_i-k_f)^2-m^2}\hat{\not e}_f u_i\right](2\pi)^4\delta^4(p_i+k_i-p_f-k_f)$$

ii) Parannihilasjon

$$S^{(ii)}=\left(\frac{1}{2\pi}\right)^6\sqrt{\frac{m^2}{E_+E_-}}\ \frac{-ie^2}{2\sqrt{\omega_1\omega_2}}\Big[\int d^4p\ \bar{v}_+\gamma^\nu\hat{e}_{2\nu}\frac{\gamma p+m}{p^2-m^2}\gamma^\mu\hat{e}_{1\mu}u_-\ \delta^4(k_2-p_+-p)\,\delta^4(k_1+p-p_-)$$

$$+\int d^4p\ \bar{v}_+\gamma^\nu\hat{e}_{1\nu}\frac{\gamma p+m}{p^2-m^2}\gamma^\mu\hat{e}_{2\mu}u_-\ \delta^4(k_1-p_+-p)\,\delta^4(k_2+p-p_-)\Big](2\pi)^4$$

$$=\left(\frac{1}{2\pi}\right)^6\sqrt{\frac{m^2}{E_+E_-}}\ \frac{-ie^2}{2\sqrt{\omega_1\omega_2}}\left[\bar{v}_+\hat{\not e}_2\frac{\not p_--\not k_1+m}{(p_--k_1)^2-m^2}\hat{\not e}_1u_-+\bar{v}_+\hat{\not e}_1\frac{\not p_--\not k_2+m}{(p_--k_2)^2-m^2}\hat{\not e}_2u_-\right](2\pi)^4\delta^4(k_1+k_2-p_--p_+)$$

c) $\psi(t) = U(t,t_0)\,\psi(t_0)$

Slutt-tilstand $\psi_f = \lim_{t\to\infty}\psi(t)$ $\qquad \psi_f = S_{fi}\psi_i \qquad S = \lim_{\substack{t\to+\infty\\ t_0\to-\infty}} U(t,t_0)$

Begynnelsestilstand $\psi_i = \lim_{t\to-\infty}\psi(t)$

Sannsynligheten for å være i ψ_f ved $t\to\infty$ når en startet i ψ_i ved $t\to-\infty$ er $|\psi_f|^2 = |S_{fi}|^2|\psi_i|^2 = |S_{fi}|^2$ da $|\psi_i|^2 = 1$

Hvis det med de gitte spesifikasjoner er flere slutt-tilstander som samme S_{fi} har en $|S_{fi}|^2 dn_f$

Overgangssannsynlighet pr tidsenhet er $\lim_{T\to\infty}\frac{|S_{fi}|^2dn_f}{T}$ $\qquad T = t-t_0$

Spredningstverrsnitt

$d\sigma = \lim_{T\to\infty}\int\frac{|S_{fi}|^2dn_f}{T\,j_{in}}$ $\qquad$ Integrert over de variable som ikke observeres.

$j_{in} = v_i/V$

Med $S_{fi} = \delta_{if} + K_{fi}(2\pi)^4\delta^4(P_f-P_i)$

og $$\left[(2\pi)^4\delta^4(P_f-P_i)\right]^2 = \lim_{\substack{T\to\infty\\V\to\infty}}\int_{VT}e^{i(P_f-P_i)x}d^4x\int_{VT}e^{i(P_f-P_i)x'}d^4x'$$

$$=\lim\int_{VT}d^4x\int_{VT}d^4(x+x')\,e^{i(P_f-P_i)(x+x')} = VT(2\pi)^4\delta^4(P_f-P_i)$$

fører

$$d\sigma = \frac{\int |K_{fi}|^2 V (2\pi)^4 \delta^4(P_f - P_i)\, dn_f}{v_i / V} = \frac{V^2}{v_i} \int |K_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i)\, dn_f$$

Med 2 partikler i slutt-tilstanden: $dn_f = \frac{V d^3 p_1}{(2\pi)^3} \frac{V d^3 p_2}{(2\pi)^3}$

$$d\sigma = \frac{(2\pi)^4 V^4}{(2\pi)^6 v_i} \int |K_{fi}|^2 \delta^4(p_1 + p_2 - P_i)\, d^3p_1\, d^3p_2 = \frac{V^4}{(2\pi)^2 v_i} \int |K_{fi}|^2_{\vec{p}_2 = \vec{P}_i - \vec{p}_1} \delta(E_f - E_i)\, d^3 p_1$$

$$\frac{d\sigma}{d\Omega_f} = \frac{V^4}{(2\pi)^2 v_i} \int |K_{fi}|^2 \delta(E_f(p_1) - E_i) \frac{p_1^2\, dp_1}{dE_f} dE_f = \frac{2\pi}{v_i} \left| V^2 K_{fi} \right|^2 \rho_f(E_f)$$

med $\rho_f = \frac{p_1^2\, dp_1}{(2\pi)^3 dE_f}$ og hvor spredningsretningen er gitt ved partikkel 1's retning $d\Omega_f = d\Omega_1$

d) Forenkle uttrykket for $S^{(i)}$

Ser på $(\not p_i + \not k_i + m) \not e_i\, u_i = + \not e_i (-\not p_i - \not k_i + m) u_i + 2(p_i + k_i) \hat{e}_i\, u_i$

ved å benytte $\not a \not b = -\not b \not a + 2 a^\mu b_\mu$

Benytter projeksjonsoperatoren $(-\not p + m) u = 2m \Lambda_- u = 0$

og lysets transversalitet $k_i \hat{e}_i = -\vec{k}_i \vec{e}_i = 0$ $\left(\hat{e}_i = (0, \vec{e}_i) \right)$

og at $\vec{p}_i = 0$ i lab.systemet $p_i \hat{e}_i = -\vec{p}_i \vec{e}_i = 0$

Det gir: $(\not p_i + \not k_i + m) \not e_i\, u_i = - \not e_i \not k_i\, u_i$

og tilsvarende $(\not p_i - \not k_f + m) \not e_f\, u_i = \not e_f \not k_f\, u_i$

og vi får

$$S^{(i)} = \left(\frac{1}{2\pi}\right)^6 \sqrt{\frac{m^2}{E_i E_f}} \frac{-ie^2}{2\sqrt{\omega_i \omega_f}} \left[\bar{u}_f \left(\frac{-\not e_f \not e_i \not k_i}{2 p_i k_i} - \frac{\not e_i \not e_f \not k_f}{2 p_i k_f} \right) u_i \right] (2\pi)^4 \delta^4(p_i + k_i - p_f - k_f)$$

Har her også benyttet at

$(p_i + k_i)^2 - m^2 = p_i^2 + 2 p_i k_i + k_i^2 - m^2 = 2 p_i k_i$ da $p_i^2 = E_i^2 - (\vec{p})^2 = m^2$

$(p_i - k_f)^2 - m^2 = -2 p_i k_f$ $\qquad k_i^2 = \omega_i^2 - (\vec{k})^2 = 0$