Rel. kvantem. 13.8.90

Løsninger

1 a) Kanonisk form $\quad H\Psi = [c\vec{\alpha}(\vec{P} - Q\vec{A}) + \beta Mc^2 + Q\phi] = E\Psi$

med operatorene $\quad \vec{P} = -i\hbar\nabla \qquad E = i\hbar\frac{\partial}{\partial t}$ $\qquad$ For et elektron $Q = -|e|$

$$[c\vec{\alpha}(-i\hbar\nabla - Q\vec{A}) + \beta Mc^2 + Q\phi]\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Dirac-matrisene oppfyller antikommutatorene

$$\{\alpha_i, \alpha_k\}_+ = 2\delta_{ik} \qquad \{\beta, \alpha_i\} = 0 \qquad \beta^2 = 1 \qquad i,k = 1,2,3$$

Kovariant form

$$(c\gamma^\mu(P_\mu - QA_\mu) - Mc^2)\Psi = 0 \qquad \gamma^0 = \beta \quad \vec{\gamma} = \beta\vec{\alpha} \qquad \mu,\nu = 0,1,2,3$$

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2g^{\mu\nu} \qquad g^{\mu\nu} = \begin{cases} 1 & \mu=\nu=0 \\ -1 & \mu=\nu\neq 0 \\ 0 & \mu\neq\nu \end{cases}$$

$$P_\mu = \left(\frac{E}{c}, -\vec{P}\right) = i\hbar\frac{\partial}{\partial x^\mu} \qquad A_\mu = \left(\frac{1}{c}\phi, -\vec{A}\right)$$

$$(c\gamma^\mu(i\hbar\partial_\mu - QA_\mu) - Mc^2)\Psi = 0$$

b) Fri partikkel $\quad H\Psi = (c\vec{\alpha}\vec{P} + \beta Mc^2)\Psi = E\Psi \quad , \quad E^2 = (Mc^2)^2 + (\vec{P}c)^2$

I standardrepresentasjonen $\quad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = \begin{pmatrix} u_a \\ u_b \end{pmatrix}$ 4-spinor.

$$H\Psi = \begin{pmatrix} Mc^2 & c\vec{\sigma}\vec{P} \\ c\vec{\sigma}\vec{P} & -Mc^2 \end{pmatrix} \frac{N}{(2\pi\hbar)^{3/2}} \begin{pmatrix} u_a \\ u_b \end{pmatrix} e^{\frac{i}{\hbar}(\vec{P}\vec{r} - Et)} = \frac{N}{(2\pi\hbar)^{3/2}} \begin{pmatrix} Mc^2 u_a + c\vec{\sigma}\vec{P}u_b \\ c\vec{\sigma}\vec{P}u_a - Mc^2 u_b \end{pmatrix} e^{\frac{i}{\hbar}(\vec{P}\vec{r} - Et)}$$

$$= \frac{N}{(2\pi\hbar)^{3/2}} E \begin{pmatrix} u_a \\ u_b \end{pmatrix} e^{\frac{i}{\hbar}(\vec{P}\vec{r} - Et)}$$

2 spinor

$$(E - Mc^2)u_a - c\vec{\sigma}\vec{P}u_b = 0 \qquad u_b = \frac{c\vec{\sigma}\vec{P}}{E + Mc^2}u_a \qquad u_a = \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

$$(E + Mc^2)u_b - c\vec{\sigma}\vec{P}u_a = 0$$

2 uavhengige 2-spinorer: $\chi_1^2 + \chi_2^2 = 1$ (normering) F.eks. $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u(P) = N\begin{pmatrix} \chi \\ \frac{c\vec{\sigma}\vec{P}}{E + Mc^2}\chi \end{pmatrix}$$

Normering:

$$u^\dagger u = |N|^2 \chi^\dagger \left(1 + \left(\frac{c\vec{\sigma}\vec{P}}{E + Mc^2}\right)^2\right)\chi = |N|^2 \frac{(E + Mc^2)^2 + c^2(\vec{P})^2}{(E + Mc^2)^2}\chi^\dagger\chi = |N|^2\frac{2E}{E + Mc^2} = \frac{E}{Mc^2}$$

gir $\quad N = \sqrt{\frac{E + Mc^2}{2Mc^2}} \quad$ og $\quad \underline{u(P) = \sqrt{\frac{E + Mc^2}{2Mc^2}}\begin{pmatrix} \chi \\ \frac{c\vec{\sigma}\vec{P}}{E + Mc^2}\chi \end{pmatrix}}$

For $\frac{|\vec{P}|}{Mc} \ll 1$

$$\Psi = \frac{1}{(2\pi\hbar)^{3/2}}\sqrt{\frac{E + Mc^2}{2Mc^2}}\begin{pmatrix} \chi \\ \frac{c\vec{\sigma}\vec{P}}{E + Mc^2}\chi \end{pmatrix} e^{\frac{i}{\hbar}(\vec{P}\vec{r} - Et)} \qquad E = \sqrt{(Mc^2)^2 + (\vec{P}c)^2} \approx Mc^2 + \frac{\vec{P}^2}{2M} + \dots$$

$$\approx \frac{1}{(2\pi\hbar)^{3/2}}\sqrt{\frac{2Mc^2 + \frac{\vec{P}^2}{2M} + \dots}{2Mc^2}}\begin{pmatrix} \chi \\ \frac{c\vec{\sigma}\vec{P}}{2Mc^2 + \frac{\vec{P}^2}{2M} + \dots}\chi \end{pmatrix} e^{\frac{i}{\hbar}(\vec{P}\vec{r} - Et)} \approx \underline{\frac{1}{(2\pi\hbar)^{3/2}}\left(1 + \frac{\vec{P}^2}{8M^2c^2} + \dots\right)\begin{pmatrix} \chi \\ \frac{c\vec{\sigma}\vec{P}}{2Mc^2 + \dots}\chi \end{pmatrix} e^{\frac{i}{\hbar}(\vec{P}\vec{r} - Et)}}$$

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1c) Variasjon: For $S = \int \mathcal{L}\, d^3x$ = ekstremal $\Rightarrow \frac{\delta S}{\delta \psi^+} = 0$

med $\mathcal{L} = c\bar{\psi}\left(i\hbar\gamma^\mu\partial_\mu - Mc\right)\psi$ gir Euler-Lagrange-likn. $(\bar{\psi} = \psi^+\gamma^0)$

$$\frac{\partial \mathcal{L}}{\partial \psi^+} - \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^\mu}\right)} = 0 \quad \Rightarrow \quad c\gamma^0\left(i\hbar\gamma^\mu\partial_\mu - Mc\right)\psi = 0$$

Multiplikasjon med $\frac{1}{c}\gamma^0$ gir så $\underline{\left(i\hbar c\,\gamma^\mu\partial_\mu - Mc^2\right)\psi = 0}$

Kanonisk impuls til ψ:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\hbar\bar{\psi}\gamma^0 = \underline{i\hbar\psi^+}$$

d) Kvantiseringsbetingelser for fermion felt:
Samtids antikommutatorene for de kanoniske konjugerte felt ψ og impulsene π:

$$\{\psi(\vec{r},t), \pi(\vec{r}',t)\}_+ = i\hbar\,\delta^3(\vec{r}-\vec{r}')$$
$$\{\psi(\vec{r},t), \psi(\vec{r}',t)\}_+ = 0$$
$$\{\pi(\vec{r},t), \pi(\vec{r}',t)\}_+ = 0$$

eller

$$\{\psi(\vec{r},t), \psi^+(\vec{r}',t)\}_+ = \delta^3(\vec{r}-\vec{r}')$$
$$\{\psi(\vec{r},t), \psi(\vec{r}',t)\}_+ = 0$$
$$\{\psi^+(\vec{r},t), \psi^+(\vec{r}',t)\}_+ = 0$$

e)
$$H = \int \psi^+ i\hbar\frac{\partial\psi}{\partial t}\, d^3r$$

$$= \sum_{ss'}\int d^3r\int d^3P\int d^3P' \frac{1}{(2\pi\hbar)^3}\sqrt{\frac{Mc^2\,Mc^2}{E\,E'}}\left(b^+u^+e^{\frac{i}{\hbar}(Et-\vec{P}\vec{r})} + d\,v^+e^{-\frac{i}{\hbar}(Et-\vec{P}\vec{r})}\right)E'\left(b'u'e^{-\frac{i}{\hbar}(E't-\vec{P}'\vec{r})} - d'^+v'e^{\frac{i}{\hbar}(E't-\vec{P}'\vec{r})}\right)$$

$$= \sum_{ss'}\int d^3P\int d^3P'\frac{Mc^2 E'}{E}\Big(b^+b'u^+u'\,e^{\frac{i}{\hbar}(E-E')t}\,\delta^3(\vec{P}-\vec{P}') - b^+d'^+u^+v'\,e^{\frac{i}{\hbar}(E+E')t}\,\delta^3(\vec{P}+\vec{P}')$$
$$+\, d\,b'\,v^+u'\,e^{-\frac{i}{\hbar}(E+E')t}\,\delta^3(\vec{P}+\vec{P}') - d\,d'^+v^+v'\,e^{-\frac{i}{\hbar}(E-E')t}\,\delta^3(\vec{P}-\vec{P}')\Big)$$

Benyttet: $u^+(\vec{P},s)\,u(\vec{P},s') = \frac{E}{Mc^2}\delta_{ss'}$
$u^+(\vec{P},s)\,v(-\vec{P},s') = 0$

$$= \sum_s\int d^3P\; Mc^2\left(b^+b\frac{E}{Mc^2} - 0 + 0 - d\,d^+\frac{E}{Mc^2}\right)$$

$$= \sum_s\int d^3P\; E\left(b^+b - d\,d^+\right) = \sum_s\int d^3P\; E\left(b^+b + d^+d - 1\right) \quad \text{med } E = \sqrt{(Mc^2)^2 + \vec{P}^2c^2}$$

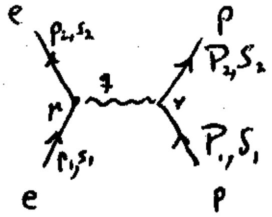
Innfører antallsoperatorer for elektroner og positroner
$N_e(P,s) = b^+(P,s)\,b(P,s)$ $\quad N_p(P,s) = d^+(P,s)\,d(P,s)$

$$H = \sum_s\int d^3P\; E\left(N_e + N_p - 1\right)$$

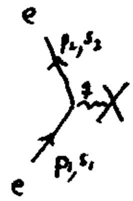
Siste ledd $\sum_s\int d^3P(-1)$ gir uendelig stor energi. Denne vakumenergien trekker vi fra ved hjelp av en normalordner som for fermioner gir $\mathcal{N}_{ord}\, d\,d^+ = -d^+d$.

$$H = \mathcal{N}_{ord}\sum_s\int d^3P\; E\left(b^+b - d\,d^+\right) = \underline{\sum_s\int d^3P\; E\left(N_e(P,s) + N_p(P,s)\right)}$$

-øsning 2 a) i)



ii)



$A^0 = \frac{e_p}{4\pi r}$, $\vec{A} = 0$

$e_p = -e$

b) i)

$$S^{(i)} = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}\cdot 4} (2\pi)^{4-4+4} \left(\frac{m}{\varepsilon_1}\frac{m}{\varepsilon_2}\frac{M}{E_1}\frac{M}{E_2}\right)^{1/2}$$

$$\int \bar{u}(p_2 s_2)(-i)e\gamma^\mu \delta^4(p_2 - p_1 - q)\frac{(-i)g_{\mu\nu}}{q^2} u(p_1,s_1)\ \bar{u}(P_2 S_2)(-i)e_p\gamma^\nu \delta^4(P_2 - P_1 + q) u(P_1,S_1)\, d^4q$$

$$S^{(i)} = \frac{-i\,e^2}{(2\pi)^2}\frac{m\,M}{(\varepsilon_1\varepsilon_2 E_1 E_2)^{1/2}} \frac{\bar{u}(p_2 s_2)\gamma^\mu u(p_1 s_1)\ \ \bar{u}(P_2 S_2)\gamma_\mu u(P_1 S_1)}{(p_1 - p_2)^2}\delta^4(p_2 + P_2 - p_1 - P_1)$$

ii)

$$S^{(ii)} = \left(\frac{1}{2\pi}\right)^{3/2\cdot 2} 2\pi \left(\frac{m}{\varepsilon_1}\frac{m}{\varepsilon_2}\right)^{1/2} \bar{u}(p_2 s_2)(-i)e\gamma^\mu A_\mu^{ytre}(\vec{p}_2 - \vec{p}_1)\,\delta(\varepsilon_1 - \varepsilon_2)\, u(p_1, s_1)$$

$$= \frac{ie^2}{(2\pi)^2}\frac{m}{(\varepsilon_1\varepsilon_2)^{1/2}}\frac{\bar{u}(p_2 s_2)\gamma^0 u(p_1, s_1)}{(\vec{p}_1 - \vec{p}_2)^2}\delta(\varepsilon_1 - \varepsilon_2) = \frac{ie^2}{(2\pi)^2}\frac{m}{(\varepsilon_1\varepsilon_2)^{1/2}}\frac{u^+(p_2 s_2)u(p_1 s_1)}{|\vec{p}_1 - \vec{p}_2|^2}\delta(\varepsilon_1 - \varepsilon_2)$$

Har benyttet at for Coulomb feltet er:

$$A_0^{ytre}(q) = \frac{+e_p}{4\pi}\int \frac{1}{r} e^{i\vec{q}\cdot\vec{r}} d^3r = \lim_{a\to 0}\frac{-e}{4\pi}\int \frac{e^{-ar}}{r} e^{iqr\cos\theta} r^2 dr\, d(\cos\theta)\, d\varphi = -\frac{e}{(\vec{q})^2}$$

$\vec{A}(q) = 0$ da $\vec{A}(\vec{r}) = 0$.

c) For lave energier $\frac{p_i}{m} \ll 1$ $\frac{P_i}{M} \ll 1$ er $\varepsilon_i \approx m$, $E_i \approx M$, $(p_1 - p_2)^2 = (\varepsilon_1 - \varepsilon_2)^2 - (\vec{p}_1 - \vec{p}_2)^2 \approx -(\vec{p}_1 - \vec{p}_2)^2$

Da $M \gg m$ vil protonet gå praktisk talt uforandret gjennom støtet $\vec{P}_1 \approx \vec{P}_2 \equiv \vec{P}$

Protonbidraget til $S^{(i)}$ vil gi

$$\bar{u}(P_2 S_2)\gamma_0 u(P_1 S_1) \approx u^+(P S_2)u(P S_1) = \delta_{S_1 S_2}$$

$$\bar{u}(P_2 S_2)\gamma_k u(P_1 S_1) \approx \left(X^+, \left(\frac{\vec{\sigma}\cdot\vec{P}}{2M}X\right)^+\right)\begin{pmatrix} & -\sigma^k \\ -\sigma^k & \end{pmatrix}\begin{pmatrix} X \\ \frac{\vec{\sigma}\cdot\vec{P}}{2M}X\end{pmatrix} =$$

$$= -\left(X^+, \left(\frac{\vec{\sigma}\cdot\vec{P}}{2M}X\right)^+\right)\begin{pmatrix}\sigma^k\frac{\vec{\sigma}\cdot\vec{P}}{2M}X \\ \sigma^k X\end{pmatrix} \approx -\left(X^+\sigma^k\frac{\vec{\sigma}\cdot\vec{P}}{2M}X + X^+\frac{\vec{\sigma}\cdot\vec{P}}{2M}\sigma^k X\right)$$

$$= -\left(X^+(\sigma^k\sigma^\ell + \sigma^\ell\sigma^k)\frac{P_\ell}{2M}X\right) = -\frac{P_k}{M}\delta_{S_1 S_2}$$

d.v.s bidraget fra γ_k-leddet er neglisjerbart i forhold til bidraget fra γ_0

$$S^{(i)} \approx \frac{-ie^2}{(2\pi)^2}\frac{m\,M}{(m^2M^2)^{1/2}}\frac{\bar{u}(p_2, s_2)\gamma^0 u(p_1 s_1)}{-(\vec{p}_1 - \vec{p}_2)^2}\delta(\varepsilon_1 - \varepsilon_2) \approx \underline{S^{(ii)}}$$

Energi bevarelsen $\delta(\varepsilon_1 - \varepsilon_2)$ sørger for at $|\vec{p}_1| = |\vec{p}_2|$ det vil si elastisk spredning. Da $M \gg m$ vil alle impuls retnings forandringer være tillatt dvs $\delta^3(\vec{p}_2 + \vec{P}_2 - \vec{p}_1 - \vec{P}_1)$ alltid oppfylt.
For lave elektron energier er altså spredning mot et proton ekvivalent med spredning mot et fast Coulomb potensial med protonladning.