Relativistisk kvantemek 29.5.92

Løsninger

1a) Kanonisk form: Lineariserer $E^2 = p^2c^2 + (mc^2)^2$ og med minimal kobling $\vec p \to \vec p - e\vec A$

$H\psi = [c\vec\alpha(\vec p - e\vec A) + \beta mc^2 + e\varphi]\psi = E\psi$ med operatorene $\vec p = -i\hbar\nabla$, $E = i\hbar\frac{\partial}{\partial t}$

Dirac-matrisene oppfyller antikommutator-reglene

$\{\alpha^i, \alpha^k\}_+ = 2\delta^{ik}$, $\{\beta, \alpha^i\} = 0$, $\beta^2 = 1$ $\quad i,k = 1,2,3$ $\quad$ Spinor: $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

$\underline{H\psi = [c\vec\alpha(-i\hbar\nabla - e\vec A) + \beta mc^2 + e\varphi]\Psi = i\hbar\frac{\partial}{\partial t}\Psi}$

Kovariant form.

Flytter over $E \to E - e\varphi$, multipliserer med β og setter $\beta\alpha^k = \gamma^k$, $\beta = \gamma^0$

$\underline{[\gamma^\mu(p_\mu - eA_\mu) - mc^2]\Psi = 0}$ med $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

b) Fri partikkel i ro $\vec p = 0$ $\quad p^0 = \frac{E}{c} = \frac{1}{c}(mc^2 + c(p)^2) = mc$, $\vec A = 0$ $\varphi = 0$

$(c\gamma^0 p^0 - mc^2)\Psi = 0$ $\quad cp^0 = i\hbar\frac{\partial}{\partial t}$ $\quad (i\hbar\frac{\partial}{\partial t} - mc^2)\psi_a = 0$

i standardrepresentasjonen: $\gamma^0 = \begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$, $\quad (-i\hbar\frac{\partial}{\partial t} - mc^2)\psi_b = 0$

med $\psi_a = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = u_a\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}mc^2t}$ $\quad u_a = \begin{pmatrix}u_1\\ u_2\end{pmatrix}$, $u_b = \begin{pmatrix}u_3\\ u_4\end{pmatrix}$ konstanter

$\psi_b = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = u_b\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{\frac{i}{\hbar}mc^2t}$

4 løsninger:

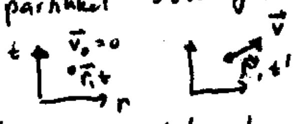
$\underline{\Psi_0^{(i)} = u^{(i)}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}E_0^{(i)}t}}$, $\quad E_0^{(i)} = \begin{cases} mc^2 & \text{for } i = 1,2 \\ -mc^2 & i = 3,4\end{cases}$

med $\underline{u_0^{(1)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u_0^{(2)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad u_0^{(3)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad u_0^{(4)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}$

$u^{(1)}$ og $u^{(3)}$ har spinnkomp. opp langs z-aksen, $u^{(2)}$ og $u^{(4)}$ ned:

$\Sigma^z u_0^{(1)(3)} = \begin{pmatrix}\sigma^z&\\&\sigma^z\end{pmatrix} u_0^{(1)(3)} = \begin{pmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{pmatrix} u_0^{(1)(3)} = + u_0^{(1)(3)}$, $\quad \Sigma^z u_0^{(2)(4)} = + u_0^{(2)(4)}$

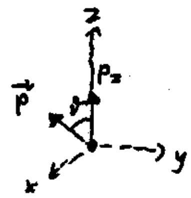
c) Sammenhengen $\Psi(\vec r', t') = S\,\Psi_0(\vec r, t)$ mellom tilstandsfunksjonen for en partikkel i ro, $\Psi_0(\vec r, t)$, og en partikkel som går med hastighet $\vec v$, $\Psi(\vec r', t')$ er bestemt av Lorentztransformasjonen mellom egensystemet L_0 og et inertialsystem som går med hastighet $-\vec v$.



Ved en Lorentz-transformasjon og dreining er 4-skalar invariant $p_\mu x^\mu = E_0 t = p_\mu' x^{\mu'} = Et' - \vec p\vec r'$

Rel. kv.mek 29.5.92

Løsn., forts.



1c forts. Oppnår partikkel med $\vec p$ i retning $\vartheta, \varphi=0$ fra partikkel i ro ved en Lorentz-transf. langs z-aksen først og så dreining om y-aksen:

$$\psi(\vec r', t') = S^D S^L \psi_0(\vec r, t)$$

For partikkelhastighet langs z-aksen er

$$\gamma^0\gamma^3 = \beta\beta\alpha^3 = \alpha^3 = \begin{pmatrix} & \sigma^3 \\ \sigma^3 & \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix}$$

og

$$S^L = \begin{pmatrix} \cosh\frac{w}{2} & & \sinh\frac{w}{2} & \\ & \cosh\frac{w}{2} & & -\sinh\frac{w}{2} \\ \sinh\frac{w}{2} & & \cosh\frac{w}{2} & \\ & -\sinh\frac{w}{2} & & \cosh\frac{w}{2} \end{pmatrix} \qquad \cosh\frac{w}{2} = \sqrt{\frac{E+mc^2}{2mc^2}}, \quad \sinh\frac{w}{2} = \frac{E+mc^2}{2mc^2}\,\frac{c\,p_z}{E+mc^2}$$

Fra partikkel i ro med spinn opp langs z-aksen (tilstand $\psi_0^{(1)}$ fig.) får da tilstandsfunksjonen for partikkel med $\vec p = (0,0,p_z)$ og positiv helisitet

$$\psi^L(\vec r_L, t_L) = S^L \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}E_0 t} = \begin{pmatrix} \cosh\frac{w}{2} \\ 0 \\ \sinh\frac{w}{2} \\ 0 \end{pmatrix} \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}(E_L t_L - \vec p_L \vec r_L)}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{c\,p_z}{E+mc^2} \\ 0 \end{pmatrix} \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}(Et - p_z z)}$$

Videre til partikkel med $\vec p = (p_x, 0, p_z)$ i retning $\vartheta, \varphi = 0$.

$$\gamma^3\gamma^1 = \beta\alpha^3\beta\alpha^1 = -\alpha^3\alpha^1 = -\begin{pmatrix} & \sigma^3 \\ \sigma^3 & \end{pmatrix}\begin{pmatrix} & \sigma^1 \\ \sigma^1 & \end{pmatrix} = -\begin{pmatrix} \sigma^3\sigma^1 & \\ & \sigma^3\sigma^1 \end{pmatrix} = -i\begin{pmatrix} \sigma^2 & \\ & \sigma^2 \end{pmatrix} = -i\Sigma^2$$

$$= \begin{pmatrix} & \sigma^3 \\ -\sigma^3 & \end{pmatrix}\begin{pmatrix} & \sigma^1 \\ -\sigma^1 & \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

$$S^D = \begin{pmatrix} \cos\vartheta/2 & -\sin\vartheta/2 & & \\ \sin\vartheta/2 & \cos\vartheta/2 & & \\ & & \cos\vartheta/2 & -\sin\vartheta/2 \\ & & \sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix}$$

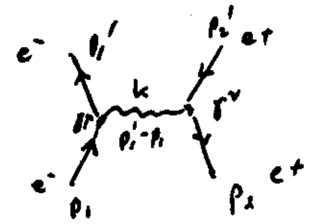
$$\psi(\vec r', t') = S^D \psi^L = \sqrt{\frac{E+mc^2}{2mc^2}} \left(\frac{1}{2\pi\hbar}\right)^{3/2} S^D \begin{pmatrix} 1 \\ 0 \\ \frac{c|\vec p|}{E+mc^2} \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar}(E_L t_L - p_L r_L)}$$

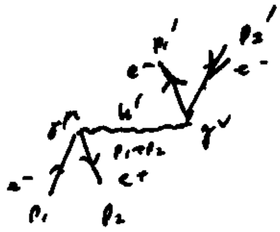
$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \cos\vartheta/2 \\ \sin\vartheta/2 \\ \frac{c|\vec p|}{E+mc^2}\cos\vartheta/2 \\ \frac{c|\vec p|}{E+mc^2}\sin\vartheta/2 \end{pmatrix} \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}(Et - \vec p \vec r)} \qquad \vec p = (p_x, 0, p_z)$$

Rel. kv.mek 29.5.97

Løsn. forts

1 d)





i siste diagram har p_1' og p_2 byttet plass så et $-$ tegn.

Bruker $\hbar = c = 1$

$$S = \left(\frac{1}{2\pi}\right)^{3/2\cdot 4} \frac{m^{1/2\cdot 4}}{(E_1'E_1\,E_2'E_2)^{1/2}} (-ie^2)^2 (2\pi)^{4\cdot 2}\frac{-i}{(2\pi)^4} \int d^4k \left(\bar{u}_1'\gamma^\mu u_1\right) \frac{g_{\mu\nu}}{k^2}\left(\bar{v}_2\gamma^\nu v_2'\right)\delta(p_1-p_1'-k)\,\delta(p_2-k-p_2')$$

$$- \left(\frac{1}{2\pi}\right)^6 \frac{m^2}{(E_1'E_1\,E_2'E_2)^{1/2}} (-e^2)(2\pi)^8\frac{-i}{(2\pi)^4} \int d^4k' \left(\bar{u}_1'\gamma^\nu v_2'\right)\frac{g_{\mu\nu}}{k'^2}\left(\bar{v}_2\gamma^\mu u_1\right)\delta(p_1+p_2-k')\,\delta(p_1'+p_2'-k')$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{m^2\, ie^2}{(E_1'E_1E_2'E_2)^{1/2}}\left[\frac{(\bar{u}_1'\gamma^\mu u_1)(\bar{v}_2\gamma_\mu v_2')}{(p_1-p_1')^2} - \frac{(\bar{u}_1'\gamma^\mu v_2')(\bar{v}_2\gamma_\mu u_1)}{(p_1+p_2)^2}\right]\delta(p_1'+p_2'-p_1-p_2)$$

1e I tyngdepunktssystemet $\vec{p}_2 = -\vec{p}_1$, $E_1 = E_2 = E$

$\vec{p}_2{}' = -\vec{p}_1{}'$, $E_1' = E_2' = E$

Setter

$\vec{p}_1$, p_2, p_2', p_1', ϑ, φ

Regner ut $\dfrac{(\bar{u}_1'\gamma^1 u_1)(\bar{v}_2\gamma_1 v_2')}{(p_1-p_1')^2}$

$\vec{p}_1'$, $\vec{p}_1'-\vec{p}_1$, ϑ, $\vec{p}_1$

$$(p_1-p_1')^2 = (E_1-E_1')^2 - (\vec{p}_1-\vec{p}_1{}')^2 = -\left(2|\vec{p}|\sin\vartheta/2\right)^2$$

e^- inn: $\vartheta = 0$ pos.helisitet (med $\hbar = c = 1$)

$$u_1 = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\0\\ \frac{|\vec{p}|}{E+m}\\0\end{pmatrix}e^{-i\varphi/2}$$

e^- ut: ϑ, φ

$$u_1' = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\cos\vartheta/2\; e^{-i\varphi/2}\\ \sin\vartheta/2\; e^{i\varphi/2}\\ \frac{|\vec{p}|}{E+m}\cos\vartheta/2\; e^{-i\varphi/2}\\ \frac{|\vec{p}|}{E+m}\sin\vartheta/2\; e^{i\varphi/2}\end{pmatrix}$$

e^+ inn: $\vartheta = \pi$, $\varphi+\pi$

$$v_2 = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\frac{-|\vec{p}|}{E+m}\\0\\1\\0\end{pmatrix}(-i)e^{-i\varphi/2}$$

e^+ ut: $\pi-\vartheta$, $\varphi+\pi$

$$v_2' = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\frac{|\vec{p}|}{E+m}\cos\vartheta/2\; ie^{-i\varphi/2}\\ \frac{|\vec{p}|}{E+m}\sin\vartheta/2\; ie^{i\varphi/2}\\ -\cos\vartheta/2\; ie^{-i\varphi/2}\\ -\sin\vartheta/2\; ie^{i\varphi/2}\end{pmatrix}$$

$$\gamma^0\gamma^1 = \begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}\begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix} = \begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix} \qquad \gamma^0\gamma_1 = -\begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix}$$

$$\bar{u}_1'\gamma^1 u_1 = u_1'^{+}\gamma\gamma u_1 = u_1'^{+}\begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix}u_1 = \frac{E+m}{2m}\left[\frac{|\vec{p}|}{E+m}\sin\vartheta/2\; e^{-i\varphi/2}e^{-i\varphi/2} + \sin\vartheta/2\; e^{-i\varphi/2}\frac{|\vec{p}|}{E+m}e^{-i\varphi/2}\right]$$

$$= \underline{\frac{|\vec{p}|}{m}\sin\vartheta/2\; e^{-i\varphi}}$$

$$\bar{v}_2\gamma_1 v_2' = -v_2^{+}\begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix}v_2' = -\frac{E+m}{2m}\left[\frac{-|\vec{p}|}{E+m}\, i e^{+i\varphi/2}(-i)\sin\vartheta/2\, e^{i\varphi/2} + (i)e^{i\varphi/2}\frac{|\vec{p}|}{E+m}\sin\vartheta/2\; ie^{i\varphi/2}\right]$$

$$= \frac{|\vec{p}|}{m}\sin\vartheta/2\; e^{i\varphi}$$

$$\frac{ie^2}{(2\pi)^2}\left(\frac{m}{E}\right)^2 \frac{\left(\frac{|\vec{p}|}{m}\right)^2\sin^2\vartheta/2\; e^{-i\varphi}e^{i\varphi}}{-4|\vec{p}|^2\sin^2\vartheta/2} = \underline{\underline{-\frac{ie^2}{(2\pi)^2\, 4E^2}}}$$

Rel. kv. mek. 29.3.92

Løsn. forsl.

2a) $\mathcal{L} = -\frac{1}{2\mu_0} K^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)(\partial_\rho A_\sigma) = -\frac{1}{2\mu_0}\left[(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\nu A^\mu)(\partial_\mu A_\nu)\right] = -\frac{1}{2\mu_0} F^{\mu\nu}\partial_\mu A_\nu$

$$= -\frac{1}{4\mu_0}\left(F^{\mu\nu}\partial_\mu A_\nu + F^{\nu\mu}\partial_\nu A_\mu\right) = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2\mu_0}(\vec E\vec D - \vec B\vec H)$$

Feltlikn:

$$0 = \frac{\partial\mathcal L}{\partial A_\lambda} - \partial_\kappa \frac{\partial \mathcal L}{\partial(\partial_\kappa A_\lambda)} = 0 - \partial_\kappa\left[-\frac{1}{2\mu_0}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma})\right]\left[\delta_{\mu\kappa}\delta_{\nu\lambda}(\partial_\rho A_\sigma) + (\partial_\mu A_\nu)\delta_{\rho\kappa}\delta_{\sigma\lambda}\right]$$

$$= \frac{1}{2\mu_0}\partial_\kappa\left[(\partial^\kappa A^\lambda - \partial^\lambda A^\kappa) + (\partial^\kappa A^\lambda - \partial^\lambda A^\kappa)\right] = \underline{\frac{1}{\mu_0}\partial_\kappa F^{\kappa\lambda} = 0}$$

De homogene Maxwells likninger følger fra formen til $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\Rightarrow \underline{\varepsilon^{\kappa\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = 0}$$

Gauge-invarians:

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ forandrer ikke verdi med ny $A^{\mu\prime} = A^\mu + \partial^\mu\chi$

hvor $\chi(\vec r,t)$ er vilkårlig funksjon!

$$\underline{F^{\mu\nu\prime}} = \partial^\mu(A^\nu + \partial^\nu\chi) - \partial^\nu(A^\mu+\partial^\mu\chi) = \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu\partial^\nu\chi - \partial^\nu\partial^\mu\chi = \underline{F^{\mu\nu}}$$

Hvis Lorentzbet. ikke oppfylt: $\partial_\mu A^\mu = B \neq 0$ så velger ny $A^{\mu\prime} = A^\mu + \partial^\mu\chi$

med χ som løsning av $\partial_\mu\partial^\mu\chi = -B$ som har løsning

Da blir $\underline{\partial_\mu A^{\mu\prime}} = \partial_\mu A^\mu + \partial_\mu\partial^\mu\chi = B - B = \underline{0}$ med samme $F^{\mu\nu}$

b) Kanoniske impulser:

$$\pi^\kappa = \frac{\partial\mathcal L}{\partial \dot A_\kappa} = \frac1c \frac{\partial \mathcal L}{\partial(\partial_0 A_\kappa)} = -\frac{1}{\mu_0 c}(\partial^0 A^\kappa - \partial^\kappa A^0) = \underline{\frac{1}{\mu_0 c}F^{\kappa 0}}$$

$\pi^0 = \frac{1}{\mu_0 c}F^{00} = 0$

$\pi^k = \frac{1}{\mu_0 c}F^{k0} = \frac{1}{\mu_0}E^k = D^k$

$$\mathcal H = \pi^\mu \dot A_\mu - \mathcal L = -\frac{1}{\mu_0 c}(\partial^0 A^\mu - \partial^\mu A^0)\,c\,\partial_0 A_\mu + \frac{1}{2\mu_0}(\partial_\nu A^\mu\partial^\nu A_\mu - \partial_\nu A^\mu \partial_\mu A^\nu)$$

$$= -\frac{1}{\mu_0}(\partial^0 A^\mu)(\partial_0 A_\mu) + \frac{1}{2\mu_0}(\partial^\nu A^\mu)(\partial_\nu A_\mu) + \frac{1}{\mu_0}(\partial^\mu A^0)(\partial_0 A_\mu) - \frac{1}{2\mu_0}(\partial^\nu A^\mu)(\partial_\mu A_\nu)$$

De to første leddene vil gi den oppgitte integranden i H

De to siste former om til totale differential ved å benytte Lorentz bet. $\partial^\nu A_\nu = 0$: $(\partial^\nu A^\mu)(\partial_\mu A_\nu) + A^\mu \underbrace{\partial_\mu\partial^\nu A_\nu}_{=0} = \partial^\nu(A^\mu\partial_\mu A_\nu)$

$$\mathcal H = -\frac{1}{2\mu_0}\partial^0 A^\mu\partial_0 A_\mu + \frac{1}{2\mu_0}\partial^k A^\mu \partial_k A_\mu + \frac1{\mu_0}\partial^\mu(A^0\partial_0 A_\mu) - \frac{1}{2\mu_0}\partial^\nu(A^\mu\partial_\mu A_\nu)$$

som gir ($\partial_k = -\partial^k$, $\partial_0 = \partial^0 = \frac1c\frac{\partial}{\partial t}$)

$$H = \int d^3x\,\mathcal H = \underline{-\frac{1}{2\mu_0}\int d^3x\left[\frac{1}{c^2}\dot A^\mu \dot A_\mu + \partial_k A^\mu\partial_k A_\mu\right]}$$

da $\int d^3x\,\partial^\mu(A^\mu\partial_\mu A_\nu) = (A^\mu\partial_\mu A)_{\text{overflate}} = 0$

· Lorentz betingelsen anvendt på plan bølge oppløsningen gir

$$\partial_\mu A^\mu = \int\frac{d^3k}{(2\pi)^{3/2}}\left(\frac{\hbar}{2\varepsilon_0\omega_k}\right)^{1/2}\left[-ik_\mu a^\mu(k)e^{-ik_\nu x^\nu} + ik_\mu a^{\mu\dagger}(k)e^{ik_\nu x^\nu}\right] = 0 \Rightarrow \underline{k_\mu a^\mu = 0}$$

For bølge i retning $\vec k$ (f.eks i z-retningen $\vec k = (0,0,k^3)$)

$k_\mu a^\mu = k^0 a^0 - \vec k\cdot\vec a = \frac{\omega_k}{c}a^0 - k a^\parallel = 0$ mens $\vec a_\perp$ er fri

(Med $\vec k = 0,0,k^3$) $\frac{\omega_k}{c}a^0 - k^3 a^3 = 0$ a^1 og a^2 er fri)

$\frac{\omega_k}{c} = |\vec k| = |k^3|$ $a^0 \mp a^3 = 0$ Feltets longitudinelle og tidskomponenter bestemmes hverandre.

Rel. kv. mek. 29.5.92
Løsn. fortsatt

2c) $$[A_\mu(x), \varepsilon_0 \dot{A}_\nu(y)]_{x_0=y_0} = \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{\varepsilon_0 \hbar (-i\omega_{k'})}{2\varepsilon_0 (\omega_k \omega_{k'})^{1/2}} \Big[[a_\mu(k), a_\nu(k')] e^{-i(kx+k'y)} + [a_\mu(k), a_\nu^+(k')] e^{-i(kx-k'y)} + [a_\mu^+(k), a_\nu(k')] e^{i(kx-k'y)} - [a_\mu^+(k), a_\nu^+(k')] e^{i(kx+k'y)} \Big]_{x_0=y_0}$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{-i\hbar}{2} \left(\frac{\omega_{k'}}{\omega_k}\right) \left[+g_{\mu\nu}\, \delta(k-k') e^{-i(kx-k'y)} + g_{\mu\nu}\, \delta(k-k')\, e^{i(kx-k'y)} \right]_{x_0=y_0}$$

$$= -\frac{i\hbar}{2} g_{\mu\nu} \left(\frac{1}{2\pi}\right)^3 \int d^3k \left[e^{-i\vec{k}(\vec{x}-\vec{y})} + e^{i\vec{k}(\vec{x}-\vec{y})} \right] \quad \text{når } x_0 = y_0$$

$$= \underline{-i\hbar\, g_{\mu\nu}\, \delta(\vec{x}-\vec{y})}$$

2d) For det kvantiserte feltet gir Lorentz-betingelsen, som generell likning for feltoperatorene $A^\mu(x)$, inkonsistens:
Ser dette ved å anvende $\frac{\partial}{\partial x_\mu}$ på kommutatoren ovenfor
Venstre side:
$$\frac{\partial}{\partial x_\mu} [A_\mu(x), \varepsilon_0 \dot{A}_\nu(y)] = [\partial^\mu A_\mu(x), \varepsilon_0 \dot{A}_\nu(y)] = 0 \quad \text{hvis } \partial_\mu A^\mu(x) = 0 \text{ generelt}$$
Høyre side:
$$-i\hbar\, g_{\mu\nu} \frac{\partial}{\partial x_\mu} \delta(\vec{x}-\vec{y}) \neq 0 \qquad \left(\int_{-\varepsilon}^{+\varepsilon} f(x)\, \delta'(x)\, dx = -f(0) \neq 0 \text{ generelt.} \right)$$
Kan ordne dette ved å forandre kommutator-regelen til bare å gjelde feltet "transversale komponenter":
Annen måte ved å innskrenke mulige felt tilstander:
Tillater bare tilstander $|\psi\rangle$ som oppfyller $\underline{\partial_\mu A^{\mu(+)}|\psi\rangle = 0}$
Her er $A^{\mu(+)}$ den del av A^μ som inneholder annihilasjons-operatorer.
Herav følger $\langle\psi|\partial_\mu A^{\mu(-)} = 0$ og dermed for alle tillatte tilstander
$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = \langle\psi|\partial_\mu A^{\mu(-)}(x) + \partial_\mu A^{\mu(+)}(x)|\psi\rangle = \langle\psi|\partial_\mu A^{\mu(-)}|\psi\rangle + \langle\psi|\partial_\mu A^{\mu(+)}|\psi\rangle = 0$$
Fra spektral-oppløsningen følger da
$$\partial_\mu A^{\mu(+)}|\psi\rangle = \int \frac{d^3k}{(2\pi)^{3/2}} \left(\frac{\hbar}{2\varepsilon_0 \omega_k}\right)^{1/2} (-ik_\mu a^\mu)\, e^{-ikx}|\psi\rangle = 0 \Rightarrow \underline{k_\mu a^\mu(k)|\psi\rangle = 0}$$
For $k = \left(\frac{\omega_k}{c}, \vec{k}\right) = \left(\frac{\omega_k}{c}, 0, 0, k^3\right) \Rightarrow \left(\frac{\omega_k}{c} a^0 - \vec{k}\vec{a}\right)|\psi\rangle = \left(\frac{\omega_k}{c} a^0 - k^3 a^3\right)|\psi\rangle = 0$

$\left(\frac{\omega_k}{c}\right)^2 = (\vec{k})^2 \Rightarrow \omega_k = ck^3 \Rightarrow \underline{(a^0 - a^3)|\psi\rangle = 0}$

Energi: For hver frekvenskomponent får vi (når vi bruker normalordning)
$$\langle\psi|H|\psi\rangle = -\frac{1}{2}\int d^3k\, \hbar\omega_k \langle\psi|(a_\mu(k) a^{\mu+}(k) + a_\mu^+(k) a^\mu(k))|\psi\rangle = -\int d^3k\, \hbar\omega_k \langle\psi|a_\mu^+(k) a^\mu(k)|\psi\rangle$$
$$\langle\psi|a_\mu^+(k) a^\mu(k)|\psi\rangle = \langle\psi|a^{0+}a^0 - a^{1+}a^1 - a^{2+}a^2 - a^{3+}a^3|\psi\rangle = -\langle\psi|a^{1+}a^1 + a^{2+}a^2|\psi\rangle$$
idet
$$\langle\psi|a^{0+}a^0 - a^{3+}a^3|\psi\rangle = \frac{1}{2}\Big(\langle\psi|(a^{0+}+a^{3+})\underbrace{(a^0-a^3)|\psi\rangle}_{=0} + \underbrace{\langle\psi|(a^{0+}-a^{3+})}_{=0}(a^0+a^3)|\psi\rangle\Big) = \underline{0}$$
Altså:
$$\underline{E = \langle\psi|H|\psi\rangle = \int d^3k\, \hbar\omega_k \langle\psi|a_1^+a_1 + a_2^+a_2|\psi\rangle \geq 0}$$ for alle tilstandene som oppfyller den modifiserte Lorentz-betingelsen $\underline{\partial_\mu A^{\mu(+)}|\psi\rangle = 0}$