Kontinuasjonseksamen i Relativistisk Kvantemekanikk

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Oppgave 1:

a)
$$\underline{\underline{i\frac{\partial}{\partial t}|\psi_I\rangle}} = i\frac{\partial}{\partial t}e^{iH_0 t}|\psi_s\rangle$$
$$= -H_0 e^{iH_0 t}|\psi_s\rangle + e^{iH_0 t}\, i\frac{\partial}{\partial t}|\psi_s\rangle$$
$$= -H_0|\psi_I\rangle + e^{iH_0 t}H|\psi_s\rangle$$
$$[H_0, V] = 0 \longrightarrow \quad = -H_0|\psi_I\rangle + He^{iH_0 t}|\psi_s\rangle$$
$$\underline{\underline{= V|\psi_I\rangle}}$$

b)
La $\left.\begin{array}{l} t_f = t_N \\ t_i = t_0 \end{array}\right\} \quad \Delta t = \dfrac{t_N - t_0}{N}$

N er stor nok til at V er konstant i intervallet Δt

$$i\frac{\partial}{\partial t}|\psi_I\rangle = i\frac{\partial}{\partial t}\,\mathcal{U}(t,t_0)|\psi_I\rangle$$
$$= V\mathcal{U}(t,t_0)|\psi_I\rangle \quad \Rightarrow$$
$$\boxed{i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = V\mathcal{U}(t,t_0)}$$

Integrere infinitesimale løsning:

$$\mathcal{U}(t_i, t_i - \Delta t) = e^{-i\Delta t V(t_i)}$$

$$\approx (1 - i\Delta t V(t_i))$$

$$\mathcal{U}(t_N, t_0) = \mathcal{U}(t_N, t_{N-1})\,\mathcal{U}(t_{N-1}, t_{N-2}) \cdots \mathcal{U}(t_1, t_0)$$

$$= (1 - i\Delta t V(t_N))(1 - i\Delta t V(t_{N-1})) \cdots$$

$$= \mathcal{T}\{(1 - i\Delta t V(t_N))(1 - i\Delta t V(t_{N-1})) \cdots \}$$

tidsordningsoperator – nå kan vi bytte om på operatorene som vi vil

$$= \mathcal{T}\Big\{ 1 + (-i)\Delta t \sum_{i=1}^{N}{}' V(t_i) + \frac{(-i)^2}{2!}\Delta t \Big(\sum_{i=1}^{N}{}' V(t_i)\Big)^2$$

$$+ \ldots = \mathcal{T} e^{-i\Delta t \sum_{i=1}^{N}{}' V(t_i)}$$

$$\Rightarrow \quad \boxed{\mathcal{U}(t_f, t_i) = \mathcal{T} e^{-i\int_{t_i}^{t_f} dt\, V(t)}}$$

c)

$$V(t) = -f(t)\,x(t)$$

$$\int_{t_i}^{t_f} V(t)\,dt = -\int_{t_i}^{t_f} f(t)\,x(t)\,dt$$

$$= -\sqrt{\frac{2}{m\omega}}\left\{ a\int_{t_i}^{t_f} dt\, f(t)\,e^{-i\omega t} + a^{+}\int_{t_i}^{t_f} dt\, f(t)\,e^{i\omega t}\right\}$$

$$= -\sqrt{\frac{2}{m\omega}}\left(a\,\tilde{f}(\omega)^{*} + a^{+}\tilde{f}(\omega)\right)$$

$$\mathcal{U}(t_f, t_i) = \mathcal{T} e^{-i\int_{t_i}^{t_f} V(t)\,dt} = \mathcal{T} e^{i\sqrt{\frac{2}{m\omega}}\left(a\tilde{f}(\omega)^{*} + a^{+}\tilde{f}(\omega)\right)}$$

$$= e^{i\sqrt{\frac{2}{m\omega}}\left(a\tilde{f}(\omega)^{*} + a^{+}\tilde{f}(\omega)\right)}$$

$\uparrow$

Siden a og a^{+} er tidsuavhengige.

d) Viser Baker-Hausdorf teoremet til 2. orden:

$$\underline{e^{A+B} = 1 + (A+B) + A^2 + AB + BA + B^2 + \cdots}$$

$$\underline{e^{A} e^{B} e^{-\frac{1}{2}[A,B]}} = \left(1 + A + \tfrac{1}{2}A^2 + \cdots\right)\left(1 + B + \tfrac{1}{2}B^2 + \cdots\right)$$
$$\left(1 - \tfrac{1}{2}[A,B] + \cdots\right)$$

$$= 1 + (A+B) + \tfrac{1}{2}A^2 + AB + \tfrac{1}{2}B^2 - \tfrac{1}{2}AB + \tfrac{1}{2}BA$$

$$\underline{= 1 + (A+B) + \tfrac{1}{2}\left(A^2 + AB + BA + B^2\right) + \cdots}$$

(4)

...t u først i 3. orden at man får
...ruk for at $[A,[A,B]] = [B,[A,B]] = 0$.

...n u en full utledning:

Vi får bruk for at

$$\underline{[A^n, B] = nA^{n-1}[A,B]}$$

som gjelder hvis $[A,[A,B]] = [B,[A,B]] = 0$

Da kan vi vise at

$$\underline{e^A B e^{-A}} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n B e^{-A}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (BA^n + [A^n, B]) e^{-A}$$

$$= B \sum_{n=0}^{\infty} \frac{1}{n!} A^n e^{-A} + \sum_{n=1}^{\infty} \frac{1}{n!} [A^n, B] e^{-A}$$

$$= B + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} A^{n-1} [A,B] e^{-A}$$

$$= B + [A,B] \sum_{n=0}^{\infty} \frac{1}{n!} A^n e^{-A}$$

$$\underline{= B + [A,B]}$$

Definér så

$$f(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$$

$$\underline{\frac{d}{d\lambda} f(\lambda)} = A e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$$

$$+ e^{\lambda A} B e^{\lambda B} e^{-\lambda(A+B)}$$

$$- e^{\lambda A} e^{\lambda B} (A+B) e^{-\lambda(A+B)}$$

$$= A f + e^{\lambda A} \cancel{e^{\lambda B} B} e^{-\lambda(A+B)}$$

$$- e^{\lambda A} e^{\lambda B} A e^{-\lambda(A+B)}$$

$$- e^{\lambda A} \cancel{e^{\lambda B} B} e^{-\lambda(A+B)}$$

$$= A f - e^{\lambda A} (A + \lambda[B, A]) e^{\lambda B} e^{-\lambda(A+B)}$$

$$e^{A} B e^{-A} = B + [A, B]$$

$$\underline{= \lambda [A, B] f(\lambda)}$$

Integrerer denne og får:

$$\underline{f(\lambda) = e^{\frac{1}{2}\lambda^2 [A, B]}}$$

⑥

Alltn°:

$$e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)} = e^{\frac{1}{2}\lambda^2 [A,B]}$$

$$\Rightarrow \quad \boxed{e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}}$$

·)

$$\underline{\mathcal{U}(t_f, t_i)} = e^{i(\underset{A}{\underset{\uparrow}{a^+\tilde{f}(\omega)}} + \underset{B}{\underset{\uparrow}{a^+\tilde{f}^*(\omega)}})}$$

$$= e^{ia^+\tilde{f}(\omega)} e^{ia\tilde{f}^*(\omega)} e^{-\frac{1}{2}[ia^+\tilde{f}(\omega),\, ia\tilde{f}^*(\omega)]}$$

$$= \underline{e^{ia^+\tilde{f}(\omega)} e^{ia\tilde{f}^*(\omega)} e^{-\frac{1}{2}|\tilde{f}(\omega)|^2}}$$

$$\left([a^+, a] = 1\right)$$

f)

$$\underline{\mathcal{U}(t_f, t_i)|0\rangle} = e^{ifa^+} e^{if^*a} |0\rangle e^{-\frac{1}{2}|f|^2}$$

$$= e^{ifa^+} |0\rangle e^{-\frac{1}{2}|f|^2}$$

$$= \underline{\sum_{k=0}^{\infty} \frac{i^k}{k!} f^k a^{+k} |0\rangle e^{-\frac{1}{2}|f|^2}}$$

$$|n\rangle = \frac{1}{\sqrt{n!}} a^{+n} |0\rangle$$

$$A_n = \langle n | U(t_f, t_i) | 0 \rangle$$

$$= e^{-\frac{1}{2}|f|^2} \sum_{k=0}^{\infty} \frac{i^k}{\sqrt{k!}} f^k \langle n | k \rangle$$

$$= \frac{e^{-\frac{1}{2}|f|^2}}{\sqrt{n!}} (if)^n$$

$$\underline{\underline{P_n = |A_n|^2 = \frac{|f|^{2n} e^{-|f|^2}}{n!}}}$$

(Poisson fordeling)

Midlere antall elektroner:

$$\underline{\underline{\langle n \rangle}} = \sum_{n=0}^{\infty} n P_n = e^{-|f|^2} \sum_{n=1}^{\infty} \frac{|f|^{2n}}{(n-1)!}$$

$$= |f|^2 e^{-|f|^2} \sum_{n=0}^{\infty} \frac{|f|^{2n}}{n!}$$

$$\underline{\underline{= |f|^2}}$$

g) Nå er

$$U(t_f, t_i) = e^{i \sum_{k,\alpha} \sqrt{\frac{1}{2\omega}} (a^+_{k\alpha} J_{k\alpha} + a_{k\alpha} J^*_{k\alpha})}$$

hvor

$$J_{k\alpha} = \sqrt{\frac{1}{V}} \int d^4x \, \epsilon_\alpha \cdot J(x) e^{ikx}$$

$$\omega = k^0 = |\vec{k}|$$

Bruker Baker-Hausdorf teoremet og at

$$[a^{+}_{k,\alpha}, a_{p,\beta}] = \delta_{kp}\,\delta_{\alpha\beta}$$

$$U(t_f,t_i) = e^{i\sum_{k\alpha}\frac{1}{\sqrt{2\omega}}a^{+}_{k\alpha}j_\alpha(k)}\, e^{i\sum_{k\alpha}\frac{1}{\sqrt{2\omega}}a_{k\alpha}j^{*}_\alpha(k)}$$

$$e^{-\frac{1}{2}\sum_{k,\alpha}\frac{|j_\alpha(k)|^2}{2\omega}}$$

I analogi med punkt f) har vi så at

$$\langle 0|U(t_f,t_i)|0\rangle = e^{-\frac{1}{2}\sum_{k,\alpha}\frac{|j_\alpha(k)|^2}{2\omega}}$$

$$\underline{\underline{P_0}} = |\langle 0|U(t_f,t_i)|0\rangle|^2$$

$$= e^{-\sum_{k,\alpha}\frac{|j_\alpha(k)|^2}{2\omega}}$$

$$= \prod_{k,\alpha} e^{-\frac{|j_\alpha(k)|^2}{2\omega}}$$

$$\langle 1_{k,\alpha}|U(t_f,t_i)|0\rangle = \langle 0|a_{k\alpha}U(t_f,t_i)|0\rangle$$

$$= \frac{i}{\sqrt{2\omega}}j_\alpha(k)\; e^{-\frac{1}{2}\sum_{k'\alpha'}\frac{|j_{\alpha'}(k')|^2}{2\omega}}$$

$$\underline{\underline{P_{1_{k\alpha}}}} = |\langle 1_{k\alpha}|U(t_f,t_i)|0\rangle|^2 = \underline{\underline{\frac{|j_\alpha(k)|^2}{2\omega}\prod_{k',\alpha'} e^{-\frac{|j_{\alpha'}(k')|^2}{2\omega'}}}}$$

Relativ sannsynlighet for emisjon av et foton:

$$\underline{\underline{p_{k\alpha}}} = \frac{P_{1k\alpha}}{P_0} = \underline{\underline{\frac{|j_\alpha(k)|^2}{2\omega}}}$$

Relativ sannsynlighet for emisjon av to fotoner med energi $k_1 \neq k_2$; $\alpha_1 \neq \alpha_2$

$$p_{k_1\alpha_1}\, p_{k_2\alpha_2}$$

Hvis $k_1 = k_2$, $\alpha_1 = \alpha_2$:

$$\frac{1}{2}\, p_{k\alpha}^{\,2}$$

↑ like partikler.

Absolutt sannsynlighet:

$$\boxed{p_{k_1\alpha_1}\, p_{k_2\alpha_2}\, P_0} \quad \text{og} \quad \boxed{\tfrac{1}{2} p_{k\alpha}^{\,2}\, P_0}$$

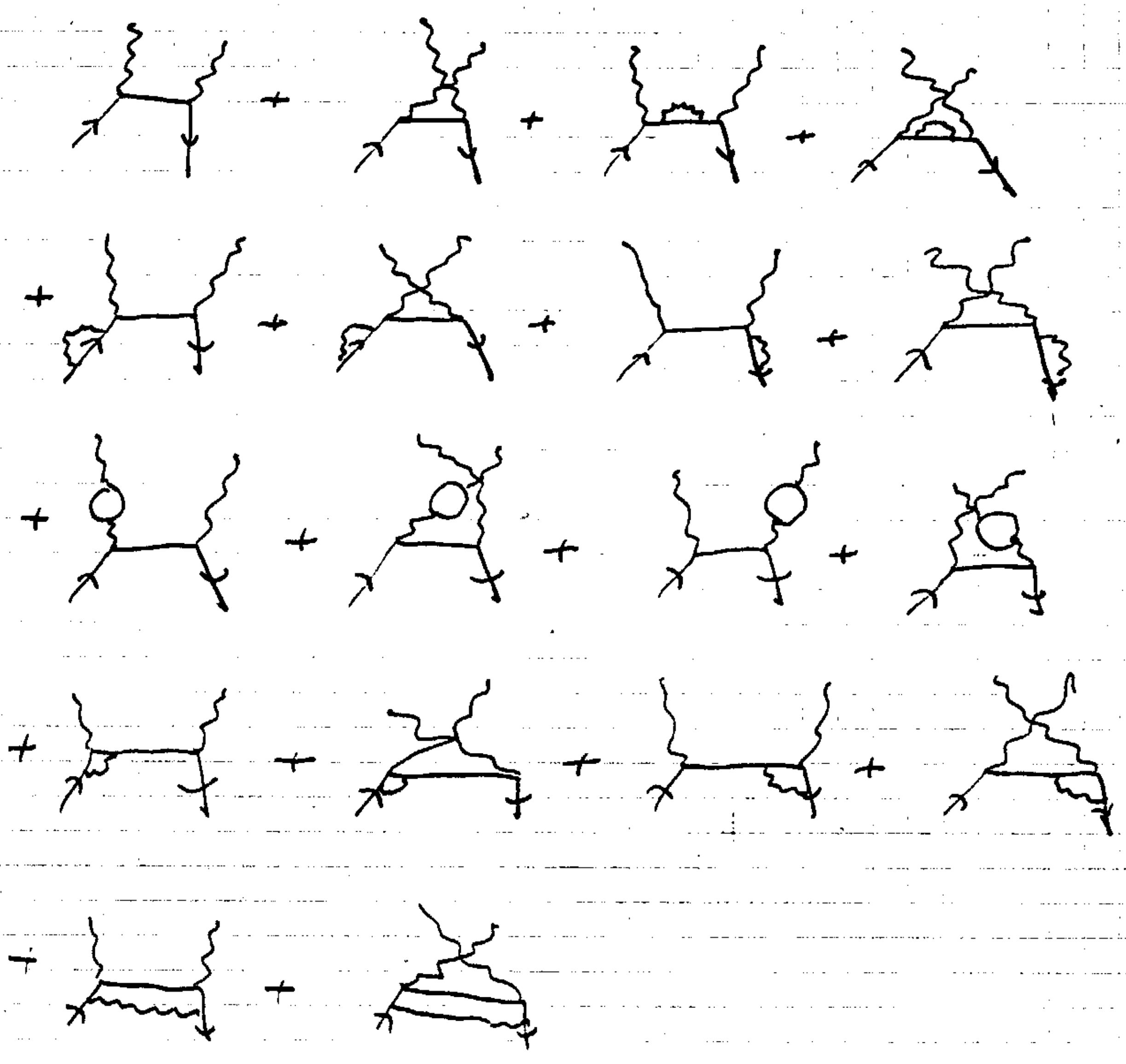
Midlere antall fotoner:

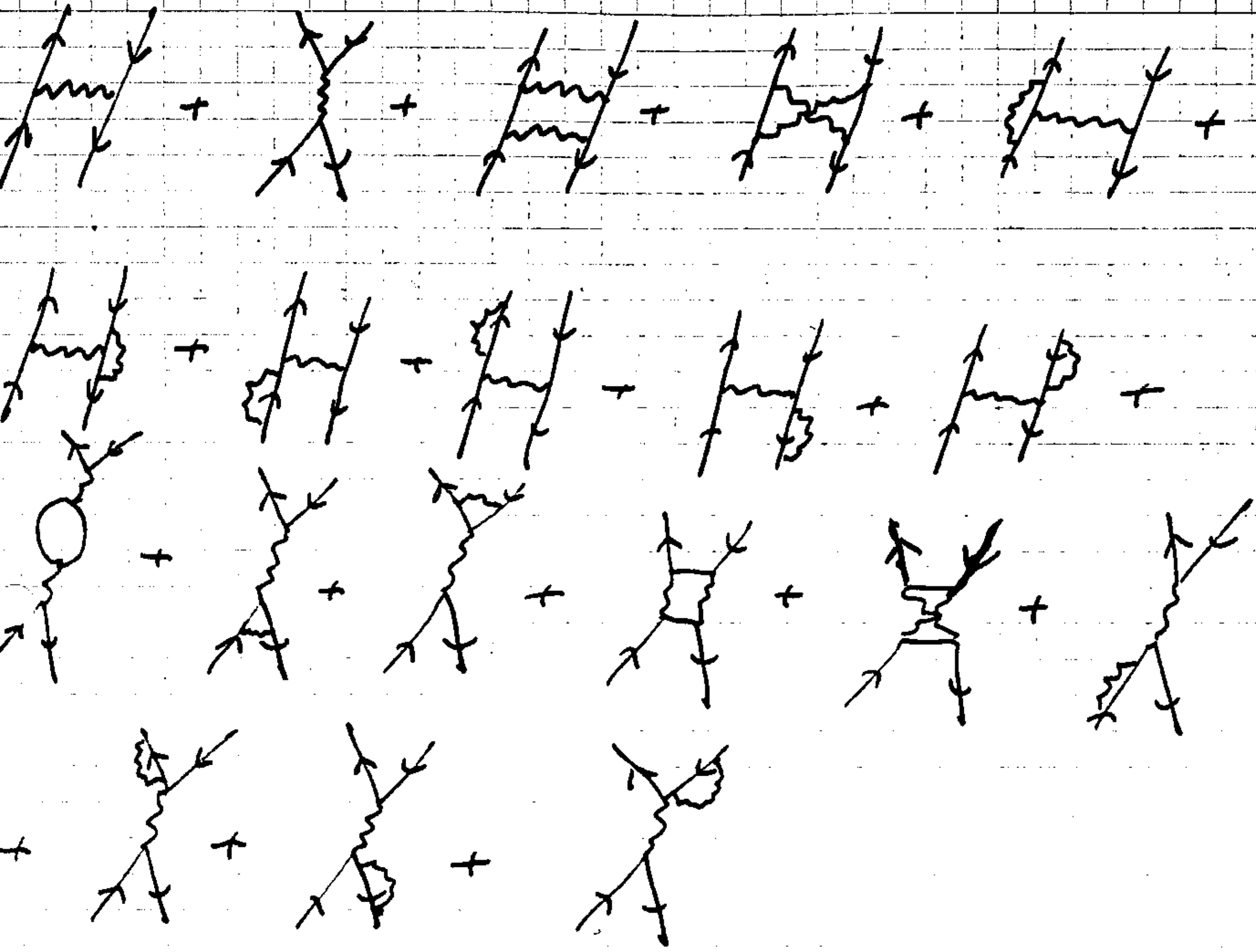
fra f) : $\langle n \rangle = |f|^2$

$$\underline{\underline{\langle n_{k\alpha} \rangle = \frac{|j_\alpha(k)|^2}{2\omega}}}$$

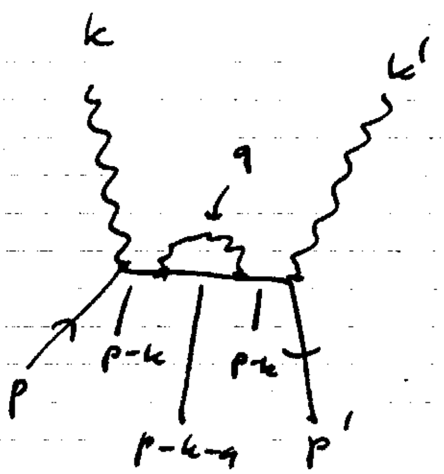
(10) <u>Opgave 2</u>

a) $e^+e^- \to \gamma\gamma$





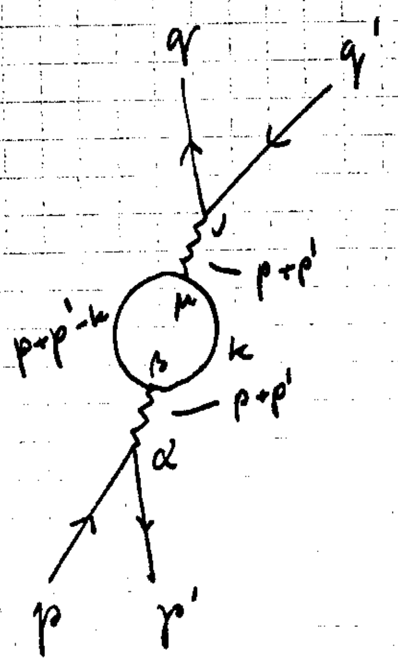
b)



$p + p' = q + q'$

$$\mathcal{M} = \bar{v}(p')\, ie \not\epsilon(k')\, iS_F(p-k) \frac{1}{(2\pi)^4}\int d^4q$$

$$\underline{iD_{F\beta\alpha}(q)\, ie\gamma^\beta\, iS_F(p-k-q)\, ie\gamma^\alpha\, iS_F(p-k)\, ie\not\epsilon(k)\, u(p)}$$



$$p+p' = q+q'$$

$$\mathcal{M} = \bar{u}(q)\, ie\gamma^{\nu} v(q')\, iD_{F\nu\mu}(p+p')$$

$$\frac{(-1)}{(2\pi)^4}\int d^4k \; \mathrm{tr}\left(iS_F(k)\, ie\gamma^{\beta}\, iS_F(p+p'-k)\, ie\gamma^{\mu} \right)$$

$$iD_{F\beta\alpha}(p+p')\, \bar{v}(p')\, ie\gamma^{\alpha} u(p)$$

c) Prosessen av typen $e^- \to e^- \gamma$ etc. har amplitude null hvis prosessen skal oppfylle impuls-konservering samtidig som at fotonet skal være masseløst.

d) Vi spesifiserte ingen dynamiske egenskaper for den klassiske strømmen i 1g, og kan derfor se bort fra impulsbevaring.