Løsningsforslag til Kontinuasjonseksamen i Relativistisk kvantemekanikk, 74327, våren 1995.

Oppgave 1

a)

Relativistiske relasjon mellom hvile-masse, m, energi, E, og impuls, $\vec{p}$:

$$E^2 = \vec{p}^{\,2} + m^2. \qquad (1)$$

Kvantemekanikk: $\left.\begin{aligned} E &\to i\frac{\partial}{\partial t} \\ \vec{p} &\to -i\vec{\nabla} \end{aligned}\right\}.$

Da blir $\left.\begin{aligned} E^2 &\to -\frac{\partial^2}{\partial t^2} \\ \vec{p}^{\,2} &\to -\nabla^2 \end{aligned}\right\}$

slik at (1) blir

$$\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 = \partial_\mu \partial^\mu + m^2 \to$$

Klein - Gordon ligningen: $\underline{(\partial_\mu \partial^\mu + m^2)\phi = 0}$

b) Ligning (2) og (3) er

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} &= \varphi \\ \frac{\partial \varphi}{\partial t} &= (\nabla^2 - m^2)\phi . \end{aligned}\right\} \quad (B)$$

(Definisjonene (4) og (5) er:

$$\left.\begin{aligned} \theta &= \frac{1}{2}\left[\phi + \frac{i}{m}\varphi\right] \\ \lambda &= \frac{1}{2}\left[\phi - \frac{i}{m}\varphi\right] . \end{aligned}\right\} \quad (C)$$

Vi løser (C) m.h.p. ϕ og φ:

$$\left.\begin{aligned} \phi &= \theta + \chi \\ \varphi &= \frac{m}{i}(\theta - \chi). \end{aligned}\right\} \quad (D).$$

Substitueres (D) i (B), framkommer

$$\left.\begin{aligned} i\frac{\partial}{\partial t}\theta &= -\frac{\nabla^2}{2m}(\theta + \lambda) + m\theta \\ i\frac{\partial}{\partial t}\lambda &= +\frac{\nabla^2}{2m}(\theta + \lambda) - m\lambda \end{aligned}\right\} \quad (E)$$

c) Antar vi $\phi = a e^{-imt}$,

har vi at

$$\theta = \frac{1}{2}\left(\phi + \frac{i}{m}\frac{\partial}{\partial t}\phi\right) = \phi$$

og

$$\chi = \frac{1}{2}\left(\phi - \frac{i}{m}\frac{\partial}{\partial t}\phi\right) = 0.$$

(Det vil si, i den ikke relativistiske grensen er

$$\chi << \theta \quad ,$$

og dermed har vi at

$$i\frac{\partial}{\partial t}\theta = -\frac{\nabla^2}{2m}(\theta + \chi) + m\theta$$

$$\longrightarrow i\frac{\partial}{\partial t}\theta = -\frac{\nabla^2}{2m}\theta + m\theta \qquad (E)$$

(Definerer vi nå $\phi = e^{-imt}\psi$, kan (E) skrives:

$$e^{-imt} i\frac{\partial}{\partial t}\psi + m e^{-imt}\psi = -e^{-imt}\frac{\nabla^2}{2m}\psi + m e^{-imt}\psi$$

Slik at
$$\underline{i\frac{\partial}{\partial t}\psi = -\frac{\nabla^2}{2m}\psi} \qquad (8)$$
som er Schrödinger ligningen.

Oppgave 2

a)

$$\underline{U = e}^{\theta\beta\vec{\alpha}\cdot\vec{p}} = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}(\beta\vec{\alpha}\cdot\vec{p})^n$$

$$= \sum_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!}(\beta\vec{\alpha}\cdot\vec{p}\,\beta\vec{\alpha}\cdot\vec{p})^n$$
$$+ \sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}\beta\vec{\alpha}\cdot\vec{p}\,(\beta\vec{\alpha}\cdot\vec{p}\,\beta\vec{\alpha}\cdot\vec{p})^n = *$$

$$\beta\vec{\alpha}\cdot\vec{p}\,\beta\vec{\alpha}\cdot\vec{p} = \beta\alpha_i\beta\alpha_j\,p_i p_j$$
$$= -\beta^2\alpha_i\alpha_j\,p_i p_j = -\alpha_i\alpha_j\,p_i p_j$$
$$= -\tfrac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i)\,p_i p_j = -p_i p_i = -\vec{p}^{\,2}$$

$$* = \sum_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!}(-1)^n|\vec{p}|^{2n}$$
$$+ \frac{\beta\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!}(-1)^n\,|\vec{p}|^{2n+1}$$

$$= \underline{\cos\theta|\vec{p}| + \frac{\beta\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\sin\theta|\vec{p}|} \qquad (14)$$

b) $\vec{\alpha}\cdot\vec{p}$ og β er hermitiske siden $H = \vec{\alpha}\cdot\vec{p} + \beta m$ er det. Det vil si $(\vec{\alpha}\cdot\vec{p})^+ = \vec{\alpha}\cdot\vec{p}$ og $\beta^+ = \beta$

(Da har vi at $\underline{(\beta\vec{\alpha}\cdot\vec{p})^+} = (\vec{\alpha}\cdot\vec{p})^+\beta^+$ (F)

$$= \vec{\alpha}\cdot\vec{p}\,\beta = \underline{-\beta\vec{\alpha}\cdot\vec{p}}$$

Bruker vi (14), har vi at

$$U^+ = \cos\theta|\vec{p}| - \frac{\beta\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\sin\theta|\vec{p}|$$

$$\underline{UU^+} = \left(\cos\theta|\vec{p}| + \frac{\beta\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\sin\theta|\vec{p}|\right)$$

$$\cdot\left(\cos\theta|\vec{p}| - \frac{\beta\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\sin\theta|\vec{p}|\right)$$

$$= \cos^2\theta|\vec{p}| - \frac{(\beta\vec{\alpha}\cdot\vec{p})^2}{\vec{p}^2}\sin^2\theta|\vec{p}|$$

$$(\beta\vec{\alpha}\cdot\vec{p})(\beta\vec{\alpha}\cdot\vec{p}) = -\vec{p}^2$$

$$= \cos^2\theta|\vec{p}| + \sin^2\theta|\vec{p}| = \underline{1}$$

Altså $\underline{U^{+} = U^{-1}}$, U er unitær.

c)

$$\underline{H'} = U H U^{-1} = \left(\cos\theta|\vec{p}| + \frac{\beta\,\vec{\alpha}\cdot\vec{p}}{|\vec{p}|} \sin\theta|\vec{p}| \right)$$

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\left(\cos\theta|\vec{p}| - \frac{\beta\,\vec{\alpha}\cdot\vec{p}}{|\vec{p}|} \sin\theta|\vec{p}| \right)$$

$$= (\vec{\alpha}\cdot\vec{p} + \beta m)\left(\cos\theta|\vec{p}| - \frac{\beta\,\vec{\alpha}\cdot\vec{p}}{|\vec{p}|} \sin\theta|\vec{p}| \right)^2$$

$$= (\vec{\alpha}\cdot\vec{p} + \beta m)\left(e^{\theta\beta\vec{\alpha}\cdot\vec{p}}\right)^2 = (\vec{\alpha}\cdot\vec{p} + \beta m)\, e^{-2\theta\beta\vec{\alpha}\cdot\vec{p}}$$

$$= (\vec{\alpha}\cdot\vec{p} + \beta m)\left(\cos 2\theta|\vec{p}| - \frac{\beta\,\vec{\alpha}\cdot\vec{p}}{|\vec{p}|} \sin 2\theta|\vec{p}| \right)$$

$$\underline{= \vec{\alpha}\cdot\vec{p}\left(\cos 2\theta|\vec{p}| - \frac{m}{|\vec{p}|} \sin 2\theta|\vec{p}| \right)}$$

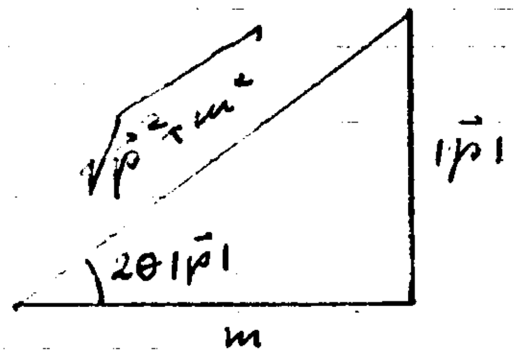
$$\underline{+ \beta\left(m\cos 2\theta|\vec{p}| + |\vec{p}|\sin 2\theta|\vec{p}| \right)} \qquad (16)$$

d) Nå velger vi θ slik at

$$\tan 2\theta|\vec{p}| = \frac{|\vec{p}|}{m} = \frac{\sin 2\theta|\vec{p}|}{\cos 2\theta|\vec{p}|}$$

Da er $\underline{H' = \beta \dfrac{m}{\cos 2\theta|\vec{p}|}}$ fra (16)

Fra figuren



har vi at $\cos 2\theta|\vec p| = \frac{m}{\sqrt{\vec p^2+m^2}}$

slik at

$$H' = \beta\frac{m}{\cos 2\theta|\vec p|} = \beta\sqrt{\vec p^2+m^2} \qquad (13)$$

e) I den vanlige representasjonen av $\beta = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

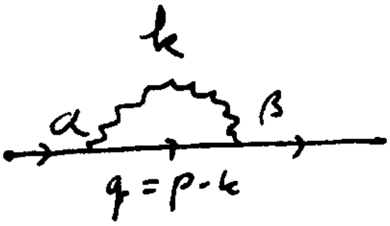
har vi at

$$H' = \begin{pmatrix}+\sqrt{\vec p^2+m^2} & 0\\ 0 & -\sqrt{\vec p^2+m^2}\end{pmatrix}. \qquad (14)$$

Begge fortegn er med.

Oppgave 3

a) Bruker Feynman reglene:



$$= i S_F(p) \, \frac{-e^2}{(2\pi)^4} \int d^4k \; i D_{F\alpha\beta}(k) \gamma^\alpha i S_F(q) \gamma^\beta i S_F(p)$$

Altså: $$\underline{\underline{-i \Sigma'(p) = \frac{-e^2}{(2\pi)^4} \int d^4k \; i D_{F\alpha\beta}(k) \gamma^\alpha i S_F(p-k) \gamma^\beta}}$$

$$i D_{F\alpha\beta}(k) = i \, \frac{-g_{\alpha\beta}}{k^2 + i\epsilon} \qquad i S_F(p-k) = \frac{i}{\not p - \not k - m + i\epsilon}$$

$$\longrightarrow \quad \underline{\underline{-i \Sigma'(p) = \frac{e^2}{(2\pi)^4} \int d^4k \; \frac{1}{k^2 + i\epsilon} \gamma_\mu \frac{1}{\not p - \not k - m + i\epsilon} \gamma^\mu}}$$

Skrevet ut eksplisitt.

b) $$\frac{1}{A+B} \cdot (A+B) = 1 = \frac{1}{A}(A+B) - \frac{1}{A} B \frac{1}{A}(A+B)$$

$$+ \frac{1}{A} B \frac{1}{A} B \frac{1}{A}(A+B) \cdots = 1 + \frac{1}{A} \cdot B - \frac{1}{A} B \left(1 + \frac{1}{A} B\right)$$

$$+ \frac{1}{A} B \frac{1}{A} B \left(1 + \frac{1}{A} B\right) - \cdots =$$

$$= 1 + \cancel{\frac{1}{A}B} - \cancel{\frac{1}{A}B} - \cancel{\frac{1}{A}B\frac{1}{A}B} + \cancel{\frac{1}{A}B\frac{1}{A}B} +$$

$$+ \frac{1}{A}B\frac{1}{A}B\frac{1}{A}B - \cdots \, .$$

Alle termen $\neq 1$ kanselleren parvis.

c).

$$\underline{\underline{iS_F'}} = iS_F + iS_F(-i\Sigma)\,iS_F + iS_F(-i\Sigma_1')\,iS_F(-i\Sigma)\,iS_F + \cdots$$

$$= i\left\{ S_F - S_F(-\Sigma)S_F + S_F(-\Sigma)S_F(-\Sigma)S_F + \cdots \right\}$$

$$= i\left\{ \frac{1}{\not p - m + i\epsilon} - \frac{1}{\not p - m + i\epsilon}(-\Sigma)\frac{1}{\not p - m + i\epsilon} + \frac{1}{\not p - m + i\epsilon}(-\Sigma)\frac{1}{\not p - m + i\epsilon}(-\Sigma)\frac{1}{\not p - m + i\epsilon} + \cdots \right\}$$

$$= \underline{\underline{\frac{i}{\not p - m - \Sigma_1'(p) + i\epsilon}}}$$

d)

$$\underline{iS_F'(p)} = \frac{i}{\not p - m - A - B(\not p - m) + i\epsilon}$$

$$= \frac{i}{(1-B)(\not p - m) - A + i\epsilon}$$

$$= \frac{i}{(1-B)(\not p - m - A + i\epsilon)} = \frac{i(1+B)}{\not p - m - A + i\epsilon}$$

$\uparrow$
$AB \simeq 0$

e) Vi følger en elektron linje gjennom et Feynman diagram. Denne består av n propagatorer og $n+1$ knuter. Hver knute representeres ved $-ie\gamma_\mu$. Altså kan de n faktorene Z_2 "spises" på de n elektronladningene. Den siste ladningen absorberer renormaliseringen av bølgefunksjonsrenormaliseringen:

Det er to ytre linjer som hver gir $\sqrt{Z_2}$, $(\sqrt{Z_2})^2 = Z_2$.

Oppgave 4

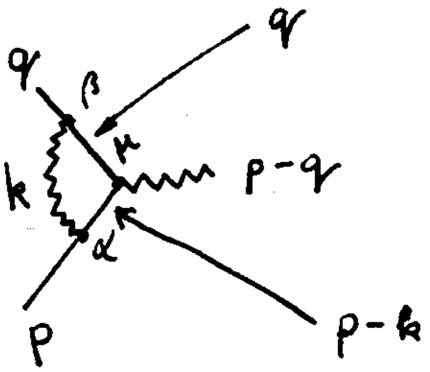
a) $$\frac{\partial}{\partial p_\mu} S_F(p) S_F^{-1}(p) = \left(\frac{\partial}{\partial p_\mu} S_F(p)\right) S_F^{-1}(p)$$

$$+ S_F(p) \frac{\partial}{\partial p_\mu} \underset{\substack{\uparrow \\ (\not p - m)}}{S_F^{-1}(p)} = \left(\frac{\partial}{\partial p_\mu} S_F(p)\right) S_F^{-1}(p) + S_F(p)\gamma^\mu$$

also:

$$\frac{\partial}{\partial p_\mu} S_F(p) + S_F(p)\gamma^\mu S_F(p) = 0$$

b)



$$ie\Lambda^\mu(p,q) =$$

$$\int \frac{d^4k}{(2\pi)^4} ie\gamma_\alpha \, iS_F(p-k)\, ie\gamma^\mu \, iS_F(q-k)\, ie\gamma_\beta \, iD_F^{\alpha\beta}(k)$$

c)

$$ie\Lambda^\mu(p,p) = ie\int \frac{d^4k}{(2\pi)^4}\, ie\gamma_\alpha \, iS_F(p-k)\gamma^\mu iS_F(p-k)\, ie\gamma_\beta \, iD_F^{\alpha\beta}(k)$$

$$= ie\int \frac{d^4k}{(2\pi)^4}\, ie\gamma_\alpha \left(-\frac{\partial}{\partial p_\mu}\right) iS_F(p-k)\, ie\gamma_\beta \, iD_F^{\alpha\beta}(k)$$

$$= -\frac{\partial}{\partial p_\mu}\, ie\int \frac{d^4k}{(2\pi)^4}\, ie\gamma_\alpha \, iS_F(p-k)\, ie\gamma_\beta \, iD_F^{\alpha\beta}(k)$$

$$= -\frac{\partial}{\partial p_\mu}\, ie\,\Sigma'(p)$$ fra opg. 3. a.