NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I
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Allowed tools: mathematical tables

1. Procca equation. ~ 25 points

A massive spin-1 particle satisfies the Procca equation,

$$(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu) A_\nu + m^2 A^\mu = 0 . \tag{1}$$

a.) "Derive" the Procca equation combining Lorentz invariance with your knowledge how many spin states a massive spin-1 particle contains.
b.) Derive the propagator $D_{\mu\nu}(k)$ of a massive spin-1 particle. [You don't have to care how the pole is handeled.]
c.) Why is the limit $m \to 0$ in your result for b.) ill-defined? [max. 50 words]
d.) Write down the generating functional $Z[J]$ for this theory.
e.) How does one obtain connected Green functions $G(x_1, \dots, x_n)$ from the generating functional $Z[J]$?

a.) Lorentz invariance requires that all four components of the free field A_μ satisfy the Klein-Gordon equation, $\left(\Box + m^2\right) A^\mu(x) = 0$. Additionally, we have to impose one constraint in order to eliminate one component. The only linear, Lorentz invariant possibility is $\partial_\mu A^\mu = 0$. To show the equivalence, act with ∂_μ on it,

$$(\partial^\nu\Box - \Box\partial^\nu) A_\nu + m^2\partial_\mu A^\mu = m^2\partial_\mu A^\mu = 0 . \tag{2}$$

Hence, a solution of the Proca equation fulfils automatically the constraint $\partial_\mu A^\mu = 0$ for $m^2 > 0$. On the other hand, we can neglect the second term in (1) for $\partial_\nu A^\nu = 0$ and obtain the Klein-Gordon equation.

b.) The propagator $D_{\mu\nu}$ is the Green function of the corresponding differential operator. Hence for a massive spin-1 field, it is determined by

$$\left[\eta^{\mu\nu}(\Box + m^2) - \partial^\mu\partial^\nu\right] D_{\nu\lambda}(x) = \delta^\mu_\lambda \delta(x) . \tag{3}$$

Performing a Fourier transformation gives

$$\left[\left(-k^2 + m^2\right)\eta^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\lambda}(k) = \delta^\mu_\lambda . \tag{4}$$

Use now the tensor method to solve this equation: In this approach, we use first all tensors available in the problem to construct the required tensor of rank 2. In the case at hand, we have at our disposal only the momentum k_μ of the particle—which we can combine to $k_\mu k_\nu$—and the metric tensor $\eta_{\mu\nu}$. Thus the tensor structure of $D_{\mu\nu}(k)$ has to be of the form

$$D_{\mu\nu}(k) = A\eta_{\mu\nu} + Bk_\mu k_\nu \tag{5}$$

with two unknown scalar functions $A(k^2)$ and $B(k^2)$. Inserting this ansatz into (4) and multiplying out, we obtain

$$\begin{aligned}\left[(-k^2+m^2)\eta^{\mu\nu}+k^\mu k^\nu\right][A\eta_{\nu\lambda}+Bk_\nu k_\lambda] &= \delta^\mu_\lambda\,,\\ -Ak^2\delta^\mu_\lambda+Am^2\delta^\mu_\lambda+Ak^\mu k_\lambda+Bm^2k^\mu k_\lambda &= \delta^\mu_\lambda\,,\\ -A(k^2-m^2)\delta^\mu_\lambda+(A+Bm^2)k^\mu k_\lambda &= \delta^\mu_\lambda\,. \end{aligned}\tag{6}$$

In the last step, we regrouped the LHS into the two tensor structures δ^μ_λ and $k^\mu k_\lambda$. A comparison of their coefficients gives then $A = -1/(k^2-m^2)$ and

$$B = -\frac{A}{m^2} = \frac{1}{m^2(k^2-m^2)}.$$

Thus the massive spin-1 propagator follows as

$$D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu}+k^\mu k^\nu/m^2}{k^2-m^2+\mathrm{i}\varepsilon}\,. \tag{7}$$

Alternative: Rewrite the Lagrange density for the Procca field as,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu A^\mu = -\frac{1}{2}A^\mu D_{\mu\nu}A^\nu \tag{8}$$

see sec. 7.3.1 of the notes for details.

c.) A massless spin-1 particle couples to a conserved current, $\partial_\mu J^\mu(x) = 0$ or $k_\mu J^\mu(k) = 0$. Technically, this means that the B term becomes undefined and the procedure fails.
More physically, we know that a massless spin-1 particle is tranverse. Thus the corresponding operator in (3) for a massless particle is a projection operator which has one eigenvalue zero corresponding to the longitudinal direction. However, a matrix with zero eigenvalues cannot be inverted.

d.) With $DA_\mu \equiv DA_0\cdots DA_3$ it is

$$Z[J_\mu] = \int \mathcal{D}A_\mu\, \exp\{\mathrm{i}\int \mathrm{d}^4x(\mathscr{L}(x)+J_\mu A^\mu\} = \mathrm{e}^{\mathrm{i}W[J]} \tag{9}$$

where $\mathscr{L}$ is given by (8).

e.) The generating functional for connected n-point functions $G(x_1,\ldots,x_n)$ is $W[J]$,

$$G(x_1,\ldots,x_n) = \frac{1}{\mathrm{i}^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\mathrm{i}W[J]\bigg|_{J=0}\,. \tag{10}$$

2. Gauge invariance. ∼ 17 points

Consider a local gauge transformation

$$U(x) = \exp[\mathrm{i}g \sum_{a=1}^{m} \vartheta^a(x) T^a] \tag{11}$$

which changes a vector of fermion fields $\boldsymbol{\psi}$ with components $\{\psi_1, \dots, \psi_n\}$ as

$$\psi(x) \to \psi'(x) = U(x)\psi(x)\,. \tag{12}$$

Assume that U are elements of a non-abelian gauge group.

a.) Derive the transformation law of $A_\mu = A_\mu^a T^a$ under a gauge transformation. One way is to require that i) the covariant derivatives transform in the same way as ψ,

$$D_\mu\psi(x) \to [D_\mu\psi(x)]' = U(x)[D_\mu\psi(x)]\,. \tag{13}$$

and ii) that the gauge field should compensate the difference between the normal and the covariant derivative,

$$D_\mu\psi(x) = [\partial_\mu + \mathrm{i}gA_\mu(x)]\psi(x)\,. \tag{14}$$

b.) Writing down the generating functional $Z[J]$ for this theory in the same way as in 1.d) results in an ill-defined expression. Why? Which solution do you suggest? [max. 50 words]

c.) Draw the Feynman rules (only the diagrams, no specific rules like $(p^\mu - p'^\mu)\gamma_\mu \dots$, group or other factors) for this theory. (The number of diagrams depends on your suggested solution in b.))

a.) Combining both requirements gives

$$D_\mu\psi(x) \to [D_\mu\psi]' = UD_\mu\psi = UD_\mu U^{-1}U\psi = UD_\mu U^{-1}\psi'\,, \tag{15}$$

and thus the covariant derivative transforms as $D'_\mu = UD_\mu U^{-1}$. Using its definition (14), we find

$$[D_\mu\psi]' = [\partial_\mu + \mathrm{i}gA'_\mu]U\psi = UD_\mu\psi = U[\partial_\mu + \mathrm{i}gA_\mu]\psi\,. \tag{16}$$

We compare now the second and the fourth term, after having performed the differentiation $\partial_\mu(U\psi)$. The result

$$[(\partial_\mu U) + \mathrm{i}gA'_\mu U]\psi = \mathrm{i}gUA_\mu\psi \tag{17}$$

should be valid for arbitrary ψ and hence we arrive after multiplying from the right with U^{-1} at

$$A_\mu \to A'_\mu = UA_\mu U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1} = UA_\mu U^{-1} - \frac{\mathrm{i}}{g}U\partial_\mu U^{-1}\,. \tag{18}$$

Here we used also $\partial_\mu(UU^{-1}) = 0$.

b.) We should integrate only over physically different field configuration; the gauge symmetry makes the path integral ill-defined, adding a factor $\Omega \times \mathbb{R}^4 = \infty$ where Ω is the "volume" of the

gauge group.
Solution: i) Fix the gauge completely, as in the Coulomb gauge in QED; this selects a certain Lorentz frame. ii) Use a covariant gauge (e.g. R_ξ); compensate the remaining unphysical degrees of freedom by adding Faddeev-Popov ghosts.

c.) Using solution ii), the vertices shown in 7.A follow: triple gauge interactions, quartic gauge interactions, two ghost gauge interaction. Using solution i) the last vertex is absent.

3. Scale invariance. ~ 15 points

Consider a massless scalar field with ϕ^4 self-interaction,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{19}$$

in $d = 4$ space-time dimensions.
a.) Find the equation of motion for $\phi(x)$.
b.) Assume that $\phi(x)$ solves the equation of motion and define a scaled field

$$\tilde{\phi}(x) \equiv \mathrm{e}^{Da}\phi(\mathrm{e}^a x) \,, \tag{20}$$

where D is a constant. Show that the scaled field $\tilde{\phi}(x)$ is also a solution of the equation of motion, provided that the constant D is choosen appropriately.
c.) Bonus question: Argue, if the classical symmetry (20) is (not) conserved on the quantum level. [max. 50 words]

a.) Using the Lagrange equation or varying directly the action gives

$$\Box\phi + \frac{\lambda}{3!}\phi^3 = 0 \,.$$

b.) Set $y = \mathrm{e}^a x$. Then

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\mu}{\partial x^\mu}\frac{\partial}{\partial y^\mu} = \mathrm{e}^a \frac{\partial}{\partial y^\mu}$$

and $\Box_x = \mathrm{e}^{2a}\Box_y$. Then $\tilde{\phi}$ satisfies the equation of motion,

$$\Box_x\tilde{\phi} + \frac{\lambda}{3!}\tilde{\phi}^3 = \mathrm{e}^{(2+D)a}\Box_x\phi + \mathrm{e}^{3Da}\frac{\lambda}{3!}\phi^3 \stackrel{!}{=} \mathrm{e}^{3a}\left[\Box_x\phi + \frac{\lambda}{3!}\phi^3\right] \stackrel{!}{=} 0 \,.$$

if we choose $D = 1$. Thus the scalar field should scale as its "naive" dimension suggests.

c.) Bonus: We discussed in Exercise sheet 7 scale invariance and noted as requirement that the classical Lagrangian contains no dimension-full parameters (which would fix scales). But loop corrections introduce necessarily a scale (μ in DR, Λ as cutoff). As a consequence, scale invariance is broken by quantum corrections.

Remarks: 1. As alternative in b), one can check the transformation of the action; surprisingly, you find then the contraint $D = 1$ and $d = 4$.
2. If we do not assume $a =$ const., we leave Minkowski space and have to consider the scalar field in a general space-time. Then one finds that the action is invariant under this transformation with an arbitrary, postive function $a(x)$, if one adds (in $d = 4$) a coupling $-R\phi^2/6$ between ϕ and the curvature scalar R.

4. Dirac (quiz). $\sim$ 10 points

a.) Helicity of a free massive particle is invariant under Lorentz transformations:

yes ☐, no ☐

Chirality of a free massive particle is invariant under Lorentz transformations

yes ☐, no ☐

b.) Helicity of a free massive particle is a conserved quantity

yes ☐, no ☐

Chirality of a free massive particle is a conserved quantity

yes ☐, no ☐

c.) Decompose a Dirac spinor ψ_D into Majorana spinors ψ_M.

d.) The bilinear $\phi_R^\dagger \sigma^\mu \phi_R$ transforms as ... under proper Lorentz transformations, as ... under parity (where ϕ_R is a Weyl spinor).

a.) no, yes; b) yes, no

c.) A Majorana spinor satisfies$\psi_M^c = e^{i\eta}\psi_M$. Thus we can contruct the two linearly independent Majorana spinors

$$\psi_{M,1} = \frac{1}{\sqrt{2}}\left(\psi_D + \psi_D^c\right) , \tag{21}$$

$$\psi_{M,2} = \frac{1}{\sqrt{2}}\left(\psi_D - \psi_D^c\right) . \tag{22}$$

out of a Dirac spinor ψ_D, or solving for ψ_D,

$$\psi_D = \frac{1}{\sqrt{2}}\left(\psi_{M,1} + \psi_{M,2}\right)$$

d.) ...vector... negative parity/axial vecor