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Examination for FY3464 Quantum Field Theory I
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Allowed tools: mathematical tables

1. Scalar field and scale invariance.
Consider a complex, scalar field ϕ with mass m and self-interaction $g\phi^n$.
a.) Write down the Lagrange density $\mathscr{L}$, explain your choice of signs and pre-factors (when physically relevant). (5 pts)
b.) Determine the mass dimension in $d = 4$ space-time dimensions of all quantities in the Lagrange density $\mathscr{L}$. Choose n such that the coupling g is dimensionless. (5 pts)
c.) Set now $m = 0$ and consider a real scalar field ϕ. Find the equation of motion for $\phi(x)$. (4 pts)
d.) Assume that $\phi(x)$ solves the equation of motion and define a scaled field

$$\tilde{\phi}(x) \equiv \mathrm{e}^{Da}\phi(\mathrm{e}^{a}x)\,, \tag{1}$$

where D and a are constants. Show that the scaled field $\tilde{\phi}(x)$ is also a solution of the equation of motion, provided that the constant D is choosen appropriately. (6 pts)
e.) Bonus question: Argue, if the classical symmetry (1) is (not) conserved on the quantum level. [max. 50 words] (2 pts)

a.) The free Lagrangian of a real scalar field is

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The relative sign is fixed by the relativistic energy-momentum relation, the overall sign by the requirement that the Hamiltonian is bounded from below. The factor 1/2 in the kinetic energy leads to "canonically normalised" field, The factor 1/2 for the mass follows then from the relativistic energy-momentum relation. Combining two real fields into a complex one, $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$ gives

$$\mathscr{L}_0 = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi.$$

If the self-interaction is odd, both choices of sign will lead to an unstable vacuum. If the self-interaction is even, the choice $-g(\phi^\dagger\phi)^{n/2}$ will lead to positive potential energy and thus stability of the vaccum.

b.) The action $S = \int \mathrm{d}^4x\mathscr{L}$ has to be dimensionless. Thus $[\mathscr{L}] = m^4$, $[\partial_\mu\phi] = m^2$, and thus $[\phi] = m$. Thence $[m] = m$, and the coupling is dimensionless, $[\lambda] = m^0$ for $n = 4$.

c.) Using the Lagrange equation or varying directly the action gives

$$\Box\phi + \frac{\lambda}{3!}\phi^3 = 0\,.$$

d.) Set $y = \mathrm{e}^a x$. Then

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\mu}{\partial x^\mu}\frac{\partial}{\partial y^\mu} = \mathrm{e}^a \frac{\partial}{\partial y^\mu}$$

and $\Box_x = \mathrm{e}^{2a}\Box_y$. Then $\tilde{\phi}$ satisfies the equation of motion,

$$\Box_x \tilde{\phi} + \frac{\lambda}{3!}\tilde{\phi}^3 = \mathrm{e}^{(2+D)a}\Box_x \phi + \mathrm{e}^{3Da}\frac{\lambda}{3!}\phi^3 \stackrel{!}{=} \mathrm{e}^{3a}\left[\Box_x \phi + \frac{\lambda}{3!}\phi^3\right] \stackrel{!}{=} 0\,.$$

if we choose $D = 1$. Thus the scalar field should scale as its "naive" dimension suggests.

e.) Bonus: We discussed in Exercise sheet 7 scale invariance and noted as requirement that the classical Lagrangian contains no dimension-full parameters (which would fix scales). But loop corrections introduce necessarily a scale (μ in DR, Λ as cutoff). As a consequence, scale invariance is broken by quantum corrections.

Remarks: 1. As alternative in b), one can check the transformation of the action; surprisingly, you find then the contraint $D = 1$ and $d = 4$.
2. If we do not assume $a = \text{const.}$, we leave Minkowski space and have to consider the scalar field in a general space-time. Then one finds that the action is invariant under this transformation with an arbitrary, postive function $a(x)$, if one adds (in $d = 4$) a coupling $-R\phi^2/6$ between ϕ and the curvature scalar R.

2. Fermion field.
Consider a massless Dirac field ψ with Lagrangian

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\partial\!\!\!/)\psi.$$

a.) Derive the propagator $S_F(p)$ of the field ψ. [You do not have to discuss how the poles of $S_F(p)$ are treated.] (4 pts)
b.) Write down the generating functional for disconnected Green functions for this theory. (4 pts)
c.) Show that the Lagrange density $\mathscr{L}$ is invariant under global vector phase transformations $\mathrm{U_V}(1)$, $\psi \to \psi' = \mathrm{e}^{\mathrm{i}\vartheta}\psi$, and under global axial phase transformations $\mathrm{U_A}(1)$, $\psi \to \psi' = \mathrm{e}^{\mathrm{i}\vartheta\gamma^5}\psi$. (6 pts)
d.) Show that global symmetry under vector phase transformation $\mathrm{U_V}(1)$ can be made local, if a coupling to a gauge boson is added. (5 pts)
e.) Draw the divergent one-loop diagrams and determine their superficial degree of divergence D (in $d = 4$ space-time dimensions) for the theory coupled to a gauge boson. (8 pts)

a.) The Green functions of the free Dirac equation (for $m > 0$) are defined by

$$(\mathrm{i}\partial\!\!\!/ - m)S(x, x') = \delta(x - x'), \tag{2}$$

where we omit on the RHS a unit matrix in spinor space. Translation invariance implies $S(x,x') = S(x-x')$ and, performing a Fourier transformation, the Fourier components $S(p)$ have to obey

$$(\not{p} - m)S(p) = 1. \tag{3}$$

After multiplication with $\not{p} + m$ and use of $\not{a}^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}a_\mu a_\nu = a^2$, we can solve for the propagator in momentum space, After multiplication with $\not{p}+m$ and use of $\not{a}^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}a_\mu a_\nu = a^2$, we can solve for the propagator in momentum space,

$$\mathrm{i}S_F(p) = \mathrm{i}\frac{\not{p}+m}{p^2 - m^2 + \mathrm{i}\varepsilon} = \frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon}, \tag{4}$$

where the last step is only meant as a symbolical shortcut.

b.) The path integral in phase space is for zero sources

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi}]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,\bar{\psi}(\mathrm{i}\not{\partial}-m)\psi}. \tag{5}$$

Adding Grassmannian sources η and $\bar{\eta}$ gives

$$\begin{aligned} Z[\eta,\bar{\eta}] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\mathrm{d}^4x'\left[\bar{\psi}(x')A(x',x)\psi(x)+\bar{\eta}(x')\psi(x)+\bar{\psi}(x')\eta(x)\right]} = \\ &= Z[0]\ \exp\left(-\mathrm{i}\int \mathrm{d}^4x\ \mathrm{d}^4x'\ \bar{\eta}(x)S_F(x-x')\eta(x')\right). \end{aligned} \tag{6}$$

c.) For vector transformations, $\psi \to \psi' = \mathrm{e}^{\mathrm{i}\vartheta}\psi$ implies $\bar{\psi} \to \bar{\psi}' = \mathrm{e}^{\mathrm{i}\vartheta}\bar{\psi}$ and the global phases drop out. For axial transformations, it follows with $\{\gamma^5, \gamma^\mu\} = 0$,

$$\psi'(x) \to \mathrm{e}^{\mathrm{i}\phi\gamma^5}\psi(x) \quad \text{and} \quad \bar{\psi}(x) \to \bar{\psi}'(x) = (\mathrm{e}^{\mathrm{i}\phi\gamma^5}\psi(x))^\dagger\gamma^0 = \bar{\psi}(x)\mathrm{e}^{\mathrm{i}\phi\gamma^5}. \tag{7}$$

Thus $\mathscr{L}$ without mass term is invariant too.

d.) Intoduce a covariant derivative,

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}qA_\mu, \tag{8}$$

which transforms homogenously,

$$D_\mu\psi(x) \to D'_\mu\psi'(x) = \{\partial_\mu + \mathrm{i}q[A_\mu(x) - \partial_\mu\Lambda(x)]\}\exp[\mathrm{i}q\Lambda(x)]\psi(x) = \tag{9}$$

$$= \exp[\mathrm{i}q\Lambda(x)]\{\partial_\mu + \mathrm{i}qA_\mu(x)\}\psi(x) = U(x)D_\mu\psi(x). \tag{10}$$

e.) The theory corresponds to QED; see e.g. Fig. 11.3 of the notes.

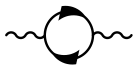
3. Unitarity.

a.) Derive the optical theorem

$$2\Im T_{ii} = \sum_n T^*_{in}T_{ni}.$$

Give a physical interpretation of this relation (less than 50 words). (6 pts)

b.) The vacuum polarisation of a photon,

q  q $\quad = \Pi^{\mu\nu}(q^2) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$

is given in dimensional regularisation by

$$\Pi(q^2) = -\frac{e^2}{12\pi^2}\left\{\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - 6\int_0^1 \mathrm{d}x\, x(1-x)\ln\left[\frac{m^2 - q^2x(1-x)}{\mu^2}\right]\right\}.$$

Show that gauge invariance, $q_\mu\Pi^{\mu\nu}(q) = 0$, implies as tensor structure of the vacuum polarisation tensor $\Pi^{\mu\nu}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$. (4 pts)

c.) Derive the imaginary part of the vacuum polarisation, $\Im[\Pi(q^2)]$. (6 pts)

d.) How does the imaginary part of the vacuum polarisation changes, if the renormalisation scheme is changed? (4 pts)

a.) The unitarity of the scattering operator, $S^\dagger S = SS^\dagger = 1$, expresses the fact that we (should) use a complete set of states for the initial and final states in a scattering process,

$$1 = \sum_n |n, +\infty\rangle\langle n, +\infty| = \sum_n S|n, -\infty\rangle\langle n, -\infty| S^\dagger = SS^\dagger. \tag{11}$$

We split the scattering operator S into a diagonal part and the transition operator T, $S = 1 + \mathrm{i}T$, and thus

$$1 = (1 + \mathrm{i}T)(1 - \mathrm{i}T^\dagger) = 1 + \mathrm{i}(T - T^\dagger) + TT^\dagger \tag{12}$$

or

$$\mathrm{i}TT^\dagger = T - T^\dagger. \tag{13}$$

We now consider matrix elements between the initial and final state,

$$\langle f| T - T^\dagger |i\rangle = T_{fi} - T_{if}^* = \mathrm{i}\langle f| TT^\dagger |i\rangle = \mathrm{i}\sum_n T_{fn}T_{in}^*. \tag{14}$$

If we set $|i\rangle = |f\rangle$, we obtain optical theorem as a connection between the forward scattering amplitude T_{ii} and the scattering into all possible states n,

$$2\Im T_{ii} = \sum_n |T_{in}|^2. \tag{15}$$

It relates the attenuation of a beam of particles in the state i, $\mathrm{d}N_i \propto -|\Im T_{ii}|^2 N_i$, to the probability that they scatter into all possible states n: what is lost, should show up somewhere.

b.) Using the tensor method, we can express $\Pi^{\mu\nu}(q^2)$ as a linear combination of $\eta^{\mu\nu}$ and $q^\mu q^\nu$. Imposing then current conservation fixes the relative sign.

c.) The only relevant part is the term with the log,

$$\Pi(q^2) = \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\ln\left[1 - \frac{q^2}{m^2}x(1-x)\right] + \dots$$

The simplest option is to find the x range for which the log is negative for a given q^2, and to use then $\Im \ln(x + \mathrm{i}\varepsilon) = -\pi$.
From $1 - \frac{q^2}{m^2}x(1-x) = 0$, it follows $x_{1/2} = \frac{1}{2} \pm \frac{1}{2}\beta$ with $\beta = \sqrt{1 - 4m^2/q^2}$. Then

$$\Im\Pi^{\mathrm{on}}(q^2 + \mathrm{i}\varepsilon) = \frac{2\alpha}{\pi}(-\pi)\int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta} \mathrm{d}x\, x(1-x) = -\frac{\alpha}{3}\beta(1 + 2m^2/q^2)\,. \tag{16}$$

c.) All the dependence on the renormalisation scheme is contained in local polynomials of the fields and their derivatives. Non-analytic functions like the log term generating the imaginary part are therefore independent of the renormalisation scheme.

Useful formulas

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{17}$$

$$\{\gamma^\mu, \gamma^5\} = 0 \quad \text{and} \quad (\gamma^5)^2 = 1\,. \tag{18}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{19}$$

$$\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 \tag{20}$$

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2}\,. \tag{21}$$

$$\frac{1}{k^2 + m^2} = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s(k^2+m^2)} \tag{22}$$

$$I_0(\omega, \alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 - m^2 + \mathrm{i}\varepsilon]^\alpha} = \mathrm{i}\,\frac{(-1)^\alpha}{(4\pi)^\omega}\frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}\,[m^2 - \mathrm{i}\varepsilon]^{\omega-\alpha}\,. \tag{23}$$

$$I(\omega, 2) = \mathrm{i}\,\frac{1}{(4\pi)^\omega}\frac{\Gamma(2-\omega)}{\Gamma(2)}\frac{1}{[m^2 - q^2 z(1-z)]^{2-\omega}}\,. \tag{24}$$

$$\Im \ln(x + \mathrm{i}\varepsilon) = -\pi \tag{25}$$