Løsningsforslag

FY 1005 / TFY 4165

22.05.2015, 09.00–13.00

Oppg 1

a : A

b : D

c : C

d : A

e : D

f : D

g : C

h : C

Oppgave 2

a)

$$Z = \sum_{\text{tilstander}} e^{-\beta E(\text{tilstander})}$$

$$= (Z_1)^N$$

Her er faktor $1/N!$ utelatt, slik at vi betrakter partiklene som klassiske og ikke ombyttbare (de kan f.eks. sitte fast i rommet)

$$\underline{\underline{Z_1 = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3}}}$$

$\varepsilon_1 = 0$; $\varepsilon_2 = \varepsilon$; $\varepsilon_3 = -\varepsilon$;

$$Z = (Z_1)^N$$

$$Z_1 = 1 + e^{-\beta\varepsilon} + e^{-\beta(-\varepsilon)}$$

$$= 1 + e^{\beta\varepsilon} + e^{-\beta\varepsilon}$$

$$\underline{\underline{= 1 + 2\cosh(\beta\varepsilon)}}$$

③

b) $\langle \varepsilon \rangle = -\frac{\partial}{\partial \beta}\left(\ln(Z_1)\right)$

$$= -\frac{\partial}{\partial \beta} \ln\left(1 + 2\cosh(\beta\varepsilon)\right)$$

$$= -\frac{1}{1+2\cosh(\beta\varepsilon)} \cdot 2\varepsilon \sinh(\beta\varepsilon)$$

$$= \underline{\underline{\frac{-2\varepsilon \sinh(\beta\varepsilon)}{1+2\cosh(\beta\varepsilon)}}}$$

$\underline{\beta \to 0}$:

$\cosh(\beta\varepsilon) \simeq 1$

$\sinh(\beta\varepsilon) \simeq \beta\varepsilon$

$$\underline{\langle \varepsilon \rangle \simeq -\frac{2}{3}\beta\varepsilon^2 \to 0}$$

Partikkelene fordelt likt på alle tre energiene.

$\beta \to \infty$:

$$\langle \varepsilon \rangle \simeq -\frac{2\varepsilon \sinh(\beta\varepsilon)}{2\cosh(\beta\varepsilon)}$$

$$= -\varepsilon \tanh(\beta\varepsilon) \simeq \underline{\underline{-\varepsilon}}$$

Alle partikler i laveste nivå

4

c) $C_V = \left(\frac{\partial u}{\partial T}\right)_V$

$$= -\frac{\partial}{\partial T}\left(\frac{2\varepsilon \sinh(\beta\varepsilon)}{1+2\cosh(\beta\varepsilon)}\right)$$

$$= -\frac{\partial}{\partial \beta}\left(\text{———}\right)\frac{\partial \beta}{\partial T}$$

$$= \frac{1}{k_B T^2}\frac{\partial}{\partial \beta}\left(\text{———}\right)$$

$$= k_B \beta^2 \frac{\partial}{\partial \beta}\left(\text{———}\right)$$

$$\frac{\partial}{\partial \beta}\left(\text{———}\right) = 2\varepsilon\left\{\frac{\varepsilon \cosh(\beta\varepsilon)}{1+2\cosh(\beta\varepsilon)} - \frac{\sinh(\beta\varepsilon)}{(1+2\cosh(\beta\varepsilon))^2}\cdot 2\varepsilon \sinh(\beta\varepsilon)\right\}$$

$$= \frac{2\varepsilon^2}{(1+2\cosh\beta\varepsilon)^2}\left\{\cosh(\beta\varepsilon) + 2\cosh^2(\beta\varepsilon) - 2\sinh^2(\beta\varepsilon)\right\}$$

$$= \frac{2\varepsilon^2}{(1+2\cosh(\beta\varepsilon))^2}\left(2+\cosh(\beta\varepsilon)\right)$$

$$C_V = 2\,\varepsilon^2 \beta^2 k_B \frac{\left(2+\cosh(\beta\varepsilon)\right)}{\left(1+2\cosh(\beta\varepsilon)\right)^2}$$

$$= 2k_B\left(\frac{\varepsilon}{k_B T}\right)^2 \left(\frac{2+\cosh(\beta\varepsilon)}{\left(1+2\cosh(\beta\varepsilon)\right)^2}\right)$$

$\beta \to \infty$

$$C_V \simeq 2(\beta\varepsilon)^2 k_B \frac{\frac{1}{2}e^{\beta\varepsilon}}{e^{2\beta\varepsilon}}$$

$$= k_B(\beta\varepsilon)^2 e^{-\beta\varepsilon} \xrightarrow{\beta\to 0} 0$$

$\beta \to \infty \quad (T \to 0)$:

Alle partiklene sånket i laveste energi-nivå, dvs grunntilstanden. Her er det ingen energi-fluktuasjoner; grunntilstanden har ingen evne til å lagre varme.

Varmekapasiteten forsvinner dermed ved lave temperaturer.

$\beta \to 0 \quad (T \to \infty)$:

$$C_V \simeq 2(\beta\varepsilon)^2 k_B \frac{3}{9} = \underline{\underline{\frac{2}{3} k_B (\beta\varepsilon)^2 \to 0}}$$

Resultatet i høytemperaturgrensen er ikke det man ville fått fra $f \neq 0$ kvadratiske frihetsgrader.

I dette tilfellet er $f = 0$, ingen kvadratiske frihetsgrader i systemet

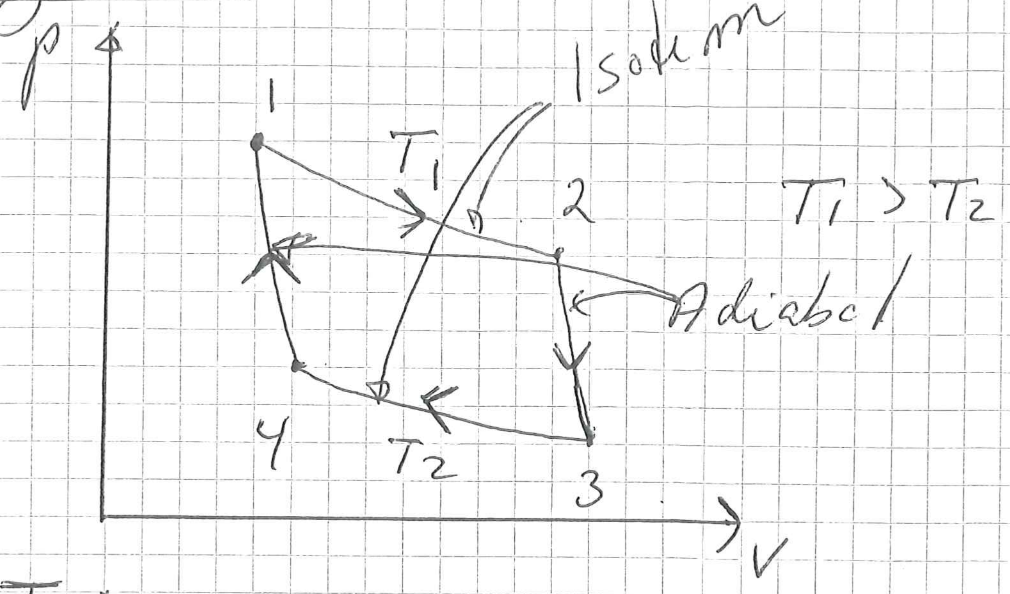
$$C_V = f \cdot \frac{1}{2} k_B$$

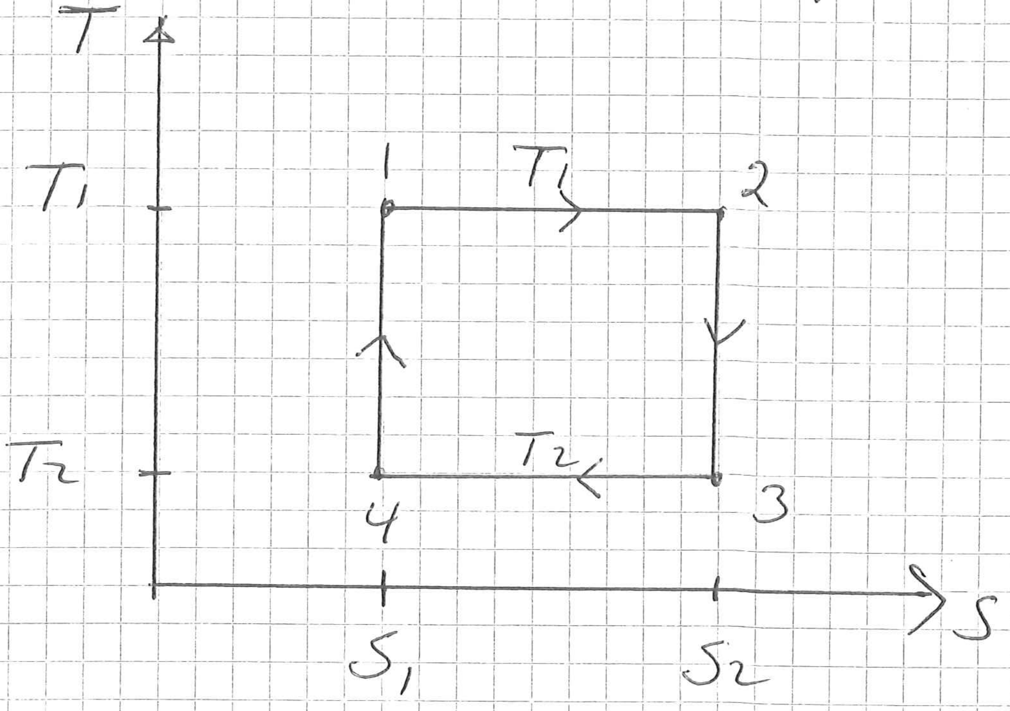
$f = 0$: $\underline{C_V = 0}$

Svaret er i overensstemmelse med ekvipartisjonsprinsippet

Oppgave 3 Carnot-prosess ⑦

a)





b) Tilført varme : Q_1 (T_1)
Avgitt varme : Q_2 (T_2)

$$\Delta Q = T\Delta S = \Delta U + p\Delta V$$

$$Q_1 = T_1 \ \Delta S \qquad (\Delta S = S_2 - S_1)$$

$$Q_2 = -T_2 \ \Delta S$$

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - |Q_2|}{Q_1} = \frac{\Delta S(T_1 - T_2)}{\Delta S \ T_1}$$

$$\eta = 1 - \frac{T_2}{T_1}$$

$$\Delta S = S_2 - S_1 = \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_T dV$$

kansellerer helt ut av η.

Kommentar:

$$T\,dS = dU + p\,dV$$

$$dS = \frac{1}{T}\left(\left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV\right) + \frac{p}{T}\,dV$$

$dT = 0$

$$dS = \frac{1}{T}\left(p + \left(\frac{\partial U}{\partial V}\right)_T\right) dV$$

$$\Delta S = \frac{1}{T_1}\int_{V_1}^{V_2}\left(p + \left(\frac{\partial U}{\partial V}\right)_{T=T_1}\right) dV$$

$$= \frac{1}{T_2}\int_{V_3}^{V_4}\left(p + \left(\frac{\partial U}{\partial V}\right)_{T=T_2}\right) dV$$

ΔS inneholder all info om tilstandsligning og termodyn. til arbeidssubstansen, men kansellerer altså ut av η.

c $1 \to 2$: Isotherm

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$\left.\begin{array}{l} dT = 0 \\ \left(\frac{\partial U}{\partial V}\right)_T = 0 \end{array}\right\} \underline{\underline{\Delta U = U(2) - U(1) = 0}}$$

$$\Delta S = \frac{1}{T_1} \int_{V_1}^{V_2} p \, dV$$

$$pV = N k_B T$$

$$\Delta S = N k_B \frac{T_1}{T_1} \int_{V_1}^{V_2} \frac{dV}{V}$$

$$\underline{\underline{= N k_B \ln\left(\frac{V_2}{V_1}\right)}}$$

$$\left(\Delta S = \cancel{N k_B} \frac{1}{T_2} \int_{V_4}^{V_3} p \, dV \right.$$

$$\left. \underline{\underline{= N k_B \ln\left(\frac{V_3}{V_4}\right)}} \right)$$

$$\left(\frac{V_3}{V_4} = \frac{V_2}{V_1} \right)$$

$$\Delta F = \Delta U - T_1 \Delta S$$

$$= 0 - N k_B T_1 \ln\left(\frac{V_2}{V_1}\right)$$

$$= - N k_B T_1 \ln\left(\frac{V_2}{V_1}\right)$$

$$V_2 > V_1 \Rightarrow \Delta F = F(2) - F(1) < 0$$

Tilstand 2 har lavest fri energi (størst entropi)

Oppgave 4

a) $\langle v \rangle$:

2D fartsfordeling

$$F_{2D}(v)\,dv = 2\pi v\, dv \left(\frac{m}{2\pi k_B T}\right) e^{-\frac{m}{2k_B T}(v_x^2+v_y^2)}$$

$$\langle v \rangle = 2\pi \frac{m}{2\pi k_B T} \underbrace{\int_0^\infty dv\, v^2 e^{-\alpha v^2}}_{= I}$$

$$\alpha = \frac{m}{2k_B T}$$

$$I = \left(-\frac{\partial}{\partial \alpha}\right)\int_0^\infty dx^2\, e^{-\alpha x^2} =$$

$$= -\frac{1}{2}\frac{\partial}{\partial \alpha}\sqrt{\frac{\pi}{\alpha}}$$

(11)

$$I = \frac{1}{4} \frac{\sqrt{\pi}}{\alpha^{3/2}}$$

$$\langle v \rangle = 2\alpha \frac{\sqrt{\pi}}{4\alpha^{3/2}}$$

$$= \sqrt{\frac{\pi}{4\alpha}} = \sqrt{\frac{2\pi \; k_B T}{4 m}}$$

$$\underline{\underline{= \sqrt{\frac{\pi}{2} \frac{k_B T}{m}}}}$$

$$\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle$$

$$\langle v^2 \rangle = \frac{m}{2\pi k_B T} \int_{-\infty}^{\infty} dv_x \int_{-\infty} dv_y \; v^2 e^{-\alpha v^2}$$

$$= \frac{2\pi m}{2\pi k_B T} \int_0^{\infty} dv \; v^3 e^{-\alpha v^2}$$

$$x = \alpha v^2$$

$$v = \frac{1}{\sqrt{\alpha}} dx$$

$$\langle v^2 \rangle = \frac{m}{k_B T} \frac{1}{\alpha^2} \int_0^{\infty} dx \; x^3 \; e^{-x^2}$$

$$z = x^2 \qquad dx\, x^3 = \frac{1}{4} dx^4$$

$$dx\, x^3 = \frac{1}{2} z\, dz$$

(12)

$$\langle v^2\rangle = \frac{m}{k_BT}\,\frac{1}{\alpha^2}\,\frac{1}{2}\int_0^\infty dz\, z\, e^{-z}$$

$$I = \int_0^\infty dz\, z\, e^{-z} = \int_0^\infty dz\, z\, e^{-\gamma z}\Big|_{\gamma=1}$$

$$= -\frac{d}{d\gamma}\int_0^\infty dz\, e^{-\gamma z}\Big|_{\gamma=1} = -\frac{d}{d\gamma}\left(\frac{1}{\gamma}\right) = \frac{1}{\gamma^2}\Big|_{\gamma=1}$$

$$\underline{\underline{I = 1}} \implies$$

$$\langle v^2\rangle = \frac{1}{2}\,\frac{2\alpha}{\alpha^2} = \frac{1}{\alpha} = \underline{\underline{\frac{2k_BT}{m}}}$$

(Kunne også gjort:

$$\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\langle v^2\rangle = 2\cdot\underbrace{\frac{1}{2}k_BT}_{v_x^2,\, v_y^2}$$

$$\langle v^2\rangle = \frac{2k_BT}{m}\quad)$$

Vi har:

$$\langle v^2\rangle - \langle v\rangle^2 = \frac{k_BT}{m}\left(2-\frac{\pi}{2}\right) > 0$$

Dette så vi generelt:

$$\langle v^2\rangle - \langle v\rangle^2 = \underline{\underline{\langle (v-\langle v\rangle)^2\rangle \geq 0}}$$

(13)

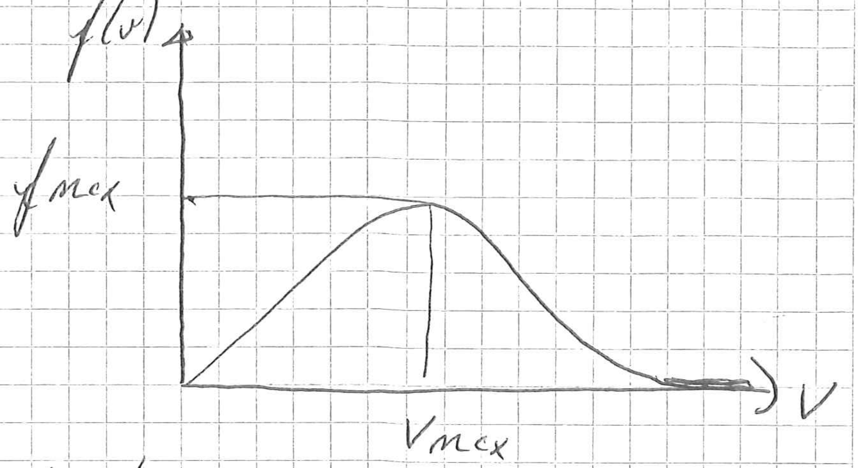
b) Fartsfordelingen:

$$f(v)\,dv = \frac{m}{2k_BT} \cdot 2\pi v\,dv\, e^{-\frac{m}{2k_BT}v^2}$$

$$f(v) = 2\pi \frac{m}{2k_BT} v\, e^{-\alpha v^2}$$

$$= C\, v\, e^{-\alpha v^2}$$

$$\alpha = \frac{m}{2k_BT}$$



V_{max}: Mest sannsynlige farten

Finnes ved:

$$\frac{df}{dv} = 0$$

$$C\left(e^{-\alpha v^2} - 2\alpha v^2 e^{-\alpha v^2}\right) = 0$$

$$1 - 2\alpha v^2 = 0$$

$$V_{max} = \sqrt{\frac{1}{2\alpha}} = \underline{\underline{\sqrt{\frac{k_BT}{m}}}}$$

(14)

$$V_{max} = \sqrt{\frac{1.38 \cdot 10^{-23} \cdot 300}{2.325 \cdot 10^{-26}}}$$

$$= \underline{\underline{422\ m/s}}$$

c) Midlere impuls:

$$\langle p_x \rangle = m \langle v_x \rangle$$

$$\langle V_x \rangle = \sqrt{\frac{m}{2\pi k_B T}} \int_{-\infty}^{\infty} dv_x \, V_x \, e^{-\frac{m}{2k_BT} V_x^2}$$

$$= 0$$

$$\underline{\underline{\langle p_x \rangle = 0}}$$

Midlere kinetisk energi:

$$\frac{1}{2} m \langle V_x^2 \rangle = \frac{1}{2} k_B T \quad \text{(Ekvipartisjons-prinsippet)}$$

$$\langle V_x^2 \rangle = \frac{k_BT}{m}$$

Alternativ:

$$\langle V_x^2 \rangle = \sqrt{\frac{m}{2\pi k_B T}} \underbrace{\int_{-\infty}^{\infty} dv_x \, V_x^2 e^{-\alpha V_x^2}}_{\frac{1}{2} \frac{\sqrt{\pi}}{\alpha^{3/2}}}$$

$$\alpha = \frac{m}{2k_BT}$$

$$\langle v_x^2 \rangle = \sqrt{\frac{m}{2\pi k_B T}} \cdot \frac{1}{2} \sqrt{\pi} \; \frac{2k_B T}{m} \sqrt{\frac{2k_B T}{m}}$$

$$= \sqrt{\frac{m}{2\pi k_B T}} \sqrt{\frac{2\pi k_B T}{m}} \; \frac{1}{2} \; \frac{2k_B T}{m}$$

$$= \frac{k_B T}{m}$$

$$\frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} k_B T$$