TFY 4165 } Termisk fysikk
FY 1005

Kont. eksamen 07.08.2017

Løsningsforslag

1a) Stasjonære forhold $\Rightarrow \vec{\nabla}\cdot\vec{j} = 0$

$$\vec{j}(r) = -\varkappa \vec{\nabla} T = j(r)\,\hat{e}_r.$$

$$j(r) = -\varkappa \frac{dT}{dr}$$

$$\dot{Q} = \oint d\vec{A}\cdot\vec{j} = 2\pi r \cdot L \cdot j(r)$$

(Her er L lengden av sylinderen)

$$\vec{\nabla}\cdot\vec{j} = 0 \Rightarrow \oint d\vec{A}\cdot\vec{j} = \dot{Q}(r) - \dot{Q}(r+dr) = 0$$

$\frac{d\dot{Q}}{dr} = 0 \Rightarrow \dot{Q}(r)$ er uavhengig av r.

b) $$\dot{Q} = 2\pi r L\, j(r) = -2\pi r L \varkappa \frac{dT}{dr}$$

$$dT = -\frac{\dot{Q}}{2\pi\varkappa L}\frac{dr}{r}$$

$$T(r) - T(R_1) = -\frac{\dot{Q}}{2\pi L\varkappa}\ln\left(\frac{r}{R_1}\right)$$

$\dot{Q}$ fastlegges ved $T(R_2) = T_2 \Rightarrow$

$$-\frac{\dot{Q}}{2\pi\varkappa L} = \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)} \Rightarrow$$

$$T(r) = T_1 + (T_2 - T_1)\,\frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}$$

c) $T_2 = 300\ K$

Vi ser på effekt-tapet pr. lengde-enhet langs sylinderen.

$$T_1 = T_2 + \frac{(\dot{Q}/L)}{2\pi\lambda}\ln\left(\frac{R_2}{R_1}\right)$$

$$= 300 + \frac{1000}{2\pi \cdot 200}\ln(6)$$

$$= 301{,}4\ K$$

2a) $$C_y = \left(\frac{\partial H}{\partial T}\right)_y$$

Oppgitt: $$\left(\frac{\partial H}{\partial y}\right)_T = T\left(\frac{\partial X}{\partial T}\right)_y - X$$

$$\left(\frac{\partial C_y}{\partial y}\right)_T = \frac{\partial^2 H}{\partial y \partial T} = \frac{\partial^2 H}{\partial T \partial y}$$

$$\frac{\partial^2 H}{\partial T \partial y} = \left(\frac{\partial X}{\partial T}\right)_y + T\left(\frac{\partial^2 X}{\partial T^2}\right)_y - \left(\frac{\partial X}{\partial T}\right)_y$$

$$= T\left(\frac{\partial^2 X}{\partial T^2}\right)_y = \left(\frac{\partial C_y}{\partial y}\right)_T \qquad \text{Q.E.D.}$$

Alternativt kan vi bruke TDI direkte:

$$TdS = C_y\, dT + T\left(\frac{\partial X}{\partial T}\right)_y dy$$

$$dS = \left(\frac{\partial S}{\partial T}\right)_y dT + \left(\frac{\partial S}{\partial y}\right)_T dy$$

$$\left(\frac{\partial S}{\partial T}\right)_y = \frac{C_y}{T} \Rightarrow \frac{\partial^2 S}{\partial y \partial T} = \frac{1}{T}\left(\frac{\partial C_y}{\partial y}\right)_T$$

$$\left(\frac{\partial S}{\partial y}\right)_T = \left(\frac{\partial X}{\partial T}\right)_y \Rightarrow \frac{\partial^2 S}{\partial T \partial y} = \left(\frac{\partial^2 X}{\partial T^2}\right)_y$$

$$\Rightarrow \quad \left(\frac{\partial C_y}{\partial y}\right)_T = T\left(\frac{\partial^2 X}{\partial T^2}\right)_y$$

b) $$X = \chi \frac{y}{T}$$

$$\left(\frac{\partial X}{\partial T}\right)_y = -\chi \frac{y}{T^2}$$

$$\left(\frac{\partial^2 X}{\partial T^2}\right)_y = 2\chi \frac{y}{T^3} \quad \Rightarrow$$

$$\left(\frac{\partial C_y}{\partial y}\right)_T = 2\chi \frac{y}{T^2}$$

$$\underline{C_y = \chi \frac{y^2}{T^2}}$$

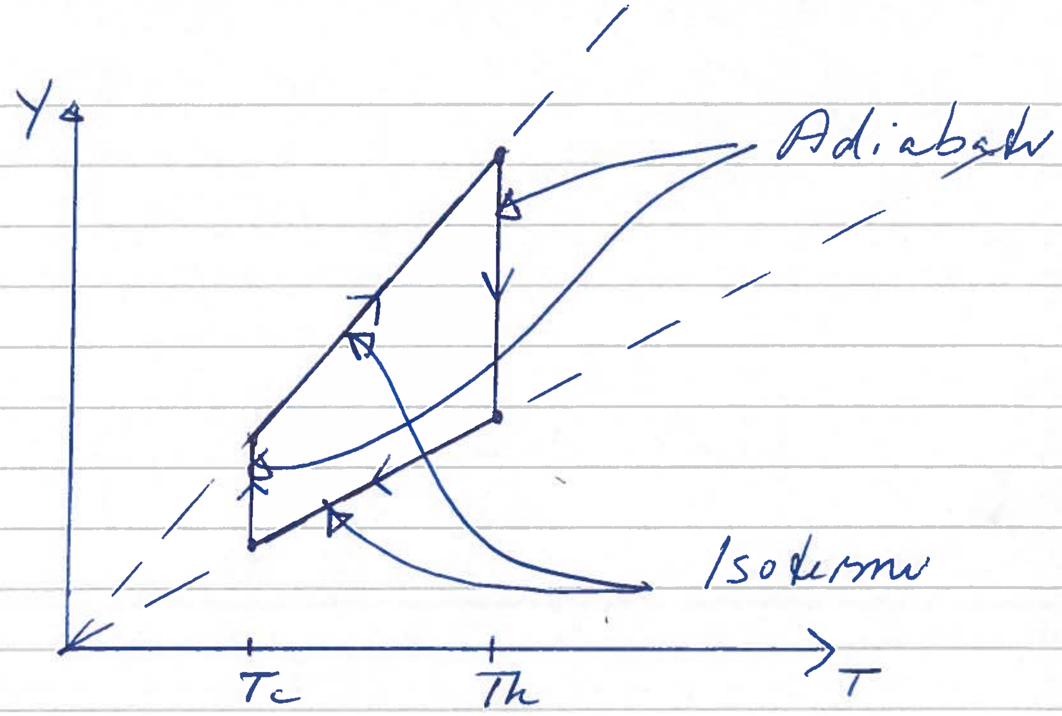
TDI langs adiabat $(TdS = 0)$

$$0 = \chi \frac{y^2}{T^2} dT - \chi \frac{y}{T} dy \quad \Big| \cdot \frac{T}{\chi y^2}$$

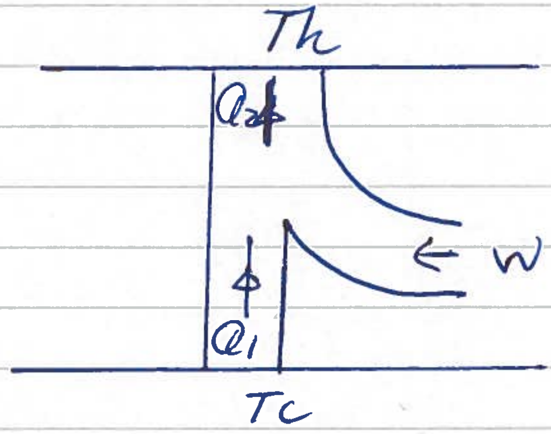
$$0 = \frac{dT}{T} - \frac{dy}{y}$$

$$\ln\left(\frac{T}{y}\right) = \text{const.} \Rightarrow \underline{\underline{\frac{y}{T} = \text{konst.}}} \quad \text{Q.E.D}$$

2c)



2d) Denne prosessen kjøres nå i revers som en varmepumpe



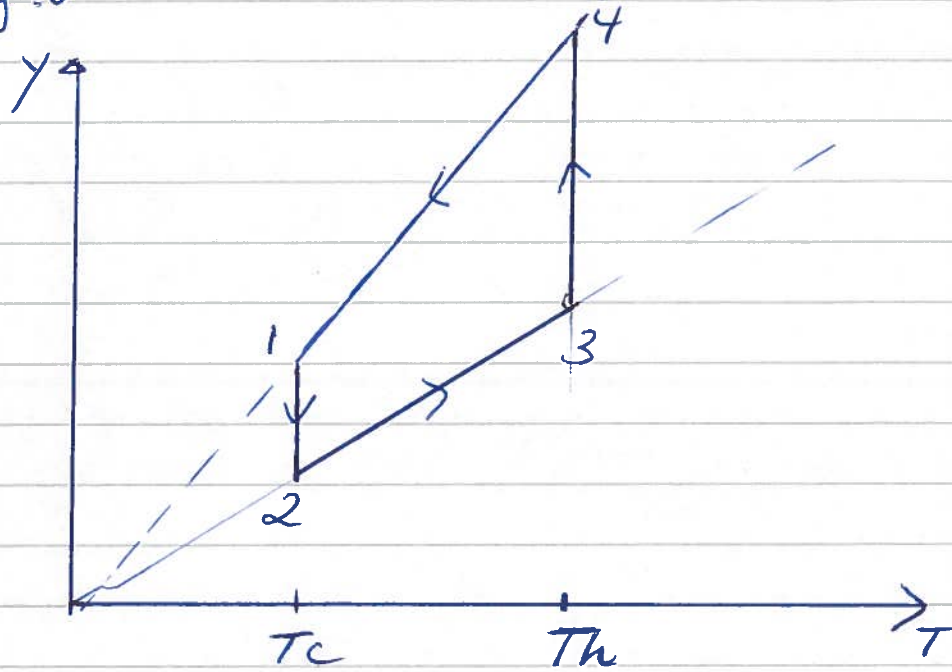
$$\eta_V = \left|\frac{Q_2}{W}\right| = \left|\frac{Q_2}{Q_1 + Q_2}\right|$$

Q_1 = varme som tilføres system ved T_c

Q_2 = varme som avgis av system ved T_h.

La oss også vise dette ved direkte utregning.

Prosess kjørt i revers



Varme tilføres ved Tc : Qt
Varme avgis ved Th : Qa

$$Q_t = \int_1^2 T\,dS = \int_1^2 T\left(\frac{\partial X}{\partial T}\right)_y dy$$

$$= -\frac{T_c}{T_c^2} X \int_1^2 y\,dy$$

$$Q_t = -\frac{X}{2T_c}\left(y_2^2 - y_1^2\right) > 0 \qquad (y_2 < y_1)$$

$$Q_a = -\frac{X}{2T_h}\left(y_4^2 - y_3^2\right) < 0 \qquad (y_4 > y_3)$$

6a

$$\eta_V = \frac{1}{\left| 1 + \frac{Q_t}{Q_a} \right|}$$

$$\frac{Q_t}{Q_a} = \frac{T_h}{T_c} \left(\frac{y_2^2 - y_1^2}{y_4^2 - y_3^2} \right)$$

Nå bruker vi at $\frac{y}{T}$ = konst. langs adiabater. Fra figur:

$$\left. \begin{aligned} \frac{y_3}{T_h} &= \frac{y_2}{T_c} \\ \frac{y_4}{T_h} &= \frac{y_1}{T_c} \end{aligned} \right\} \qquad \begin{aligned} \frac{y_3}{y_2} &= \frac{T_h}{T_c} \\ \frac{y_3}{y_4} &= \frac{y_2}{y_1} \end{aligned}$$

$$\frac{Q_t}{Q_a} = -\frac{T_h}{T_c} \left(\frac{y_2}{y_3} \right)^2 \underbrace{\frac{\left(1 - y_1^2 / y_2^2 \right)}{\left(1 - \frac{y_4^2}{y_3^2} \right)}}_{=1}$$

$$= -\frac{T_h}{T_c} \left(\frac{T_c}{T_h} \right)^2 = -\frac{T_c}{T_h}$$

$$\underline{\underline{\eta_V = \frac{1}{1 - \frac{T_c}{T_h}}}}$$

3a)

$$Z = \frac{1}{h^{2N} N!} Z_1^N$$

$$Z_1 \equiv Q_1 \cdot Z_0^2$$

$$Q_1 \equiv \int d^2r \, e^{-\frac{\beta}{2} m\omega r^2}$$

$$Z_0 = \int_{-\infty}^{\infty} dp_x \, e^{-\frac{\beta p_x^2}{2m}}$$

$$= \sqrt{\frac{2m\pi}{\beta}}$$

$$Q_1 = 2\pi \int_0^R dr \, r \, e^{-\frac{\beta}{2} m\omega^2 r^2}$$

$$= \pi \int_0^{R^2} dv \, e^{-\frac{\beta m\omega^2}{2} v}$$

$$= -\frac{2\pi}{\beta m\omega^2} \Big|_0^{R^2} e^{-\frac{\beta m\omega^2}{2} v}$$

$$Q_1 = \frac{2\pi}{\beta m\omega^2} \left(1 - e^{-\frac{\beta m\omega^2 R^2}{2}} \right)$$

$$Z = \frac{1}{h^{2N} N!}\left(2\pi m k_B T\right)^N \cdot Q_1^N$$

Termisk de Broglie bølgelengde

$$= \frac{Q_1^N}{\lambda^{2N} N!}, \qquad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

b) $$U = -\frac{\partial \ln Z}{\partial \beta}$$

$$= +2N \frac{\partial \ln \lambda}{\partial \beta} - N \frac{\partial \ln Q_1}{\partial \beta}$$

$\beta m \omega^2 R^2 \gg 1 \Rightarrow$ vi ignorerer 2. ledd i $Q_1 \Rightarrow$

$$Q_1 \simeq \frac{2\pi}{\beta m \omega^2} \Rightarrow$$

$$U = 2N \frac{\partial \ln \lambda}{\partial \beta} - N \frac{\partial \ln \left(\frac{1}{\beta}\right)}{\partial \beta}$$

$$= \frac{N}{\beta} + \frac{N}{\beta} = \underline{\underline{2N k_B T}}$$

$$\underline{\underline{C_V = 2N k_B}}$$

Klassisk ekvipartisjonsprinsipp
$\frac{k_B}{2}$ bidrag til C_v fra hver
uavh. kvadratisk frihetsgrad.
Hv: 4 kvadratiske frihetsgrader
pr. partikkel $\Rightarrow$

$$C_v = N \cdot 4 \cdot \frac{k_B}{2} = \underline{\underline{2Nk_B}}$$

c) $$p = Nk_BT \frac{\partial \ln Q_1}{\partial V}$$

$\beta m \omega^2 R^2 \gg 1 \Rightarrow Q_1$ uavh. av $R \Rightarrow$

$$\underline{p = 0}$$

d) $\beta m \omega^2 R^2 \ll 1 \Rightarrow$

$$Q_1 \simeq \frac{2\pi}{\beta m \omega^2}\left(1 - 1 - \frac{\beta m \omega^2 R^2}{2} + \ldots\right)$$

$$= \pi R^2 = V$$

$$p = Nk_BT \frac{\partial \ln V}{\partial V} = \underline{\underline{\frac{Nk_BT}{V}}}$$

Nu giver (14):

$$p = Nk_BT \frac{\partial \ln Q_1}{\partial(\pi R^2)}$$

$$= \frac{Nk_BT}{\pi} \frac{e^{-\frac{\beta m\omega^2 R^2}{2}}}{\left(1-e^{-\frac{\beta m\omega^2 R^2}{2}}\right)} \frac{\beta m\omega^2}{2}$$

$$= \frac{m\omega^2}{2\pi} \frac{e^{-\frac{\beta m\omega^2 R^2}{2}}}{1-e^{-\frac{\beta m\omega^2 R^2}{2}}} \cdot N$$

Igjen har vi at grænsen $\frac{\beta m\omega^2 R^2}{2} \ll 1 \Rightarrow$

$$p \simeq \frac{m\omega^2}{2\pi} \frac{1}{\frac{\beta m\omega^2 R^2}{2}} N$$

$$= \frac{Nk_BT}{\pi R^2} = \frac{Nk_BT}{V}$$

(11)

For fullstendighetens skyld tar vi også med det presise uttrykket i motsatt grense: $\frac{\beta m\omega^2 R^2}{2} \gg 1$

$$p \simeq N \frac{m\omega^2}{2\pi} e^{-\frac{\beta m\omega^2 R^2}{2}}$$

Som vist at trykket partiklene utøver mot veggen forsvinner eksponentielt med volumet $V = \pi R^2$ av beholderen.

Grunnen til at vi får det velkjente ideell-gass resultatet i grensen $\frac{\beta m\omega^2 R^2}{2} \ll 1$, er at i denne grensen vil partiklene være så nære origo hele tiden at effekten av det harmoniske potensialet ikke merkes.