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Oppgave 1

a) $\vec{\nabla} \cdot \vec{j} = 0 \Rightarrow \oiint d\vec{A} \cdot \vec{j} = 0$

$$P(r) = 4\pi r^2 j_r(r)$$

$$-P(r) + P(r+dr) = 0 \Rightarrow \frac{dP}{dr} = 0$$

<u>$P(r)$ uavh. av r</u>

$$P = 4\pi r^2 j_r = -4\pi \varkappa r^2 \frac{dT}{dr}$$

$$dT = -\frac{P}{4\pi\varkappa} \frac{dr}{r^2}$$

Innerste kuleskall: $R_1 < r < R_2$

$$T(r) - T_1 = \frac{-P}{4\pi\varkappa_1}\left(\frac{1}{R_1} - \frac{1}{r}\right)$$

Ytterste kuleskall: $R_1 < r < R_3$

$$T(r) - T_2 = -\frac{P}{4\pi\varkappa_2}\left(\frac{1}{R_2} - \frac{1}{r}\right)$$

$r \to R_2^-$:

$$T_2 - T_1 = -\frac{P}{4\pi\varkappa_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$P = G_1\left(T_1 - T_2\right) \Rightarrow$$

$$\underline{\underline{\frac{1}{G_1} = \frac{1}{4\pi\varkappa_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}}$$

$r \to R_3^-$:

$$T_3 - T_2 = -\frac{P}{4\pi\varkappa_2}\left(\frac{1}{R_2}-\frac{1}{R_3}\right)$$

$$P = G_2\left(T_2 - T_3\right) \Rightarrow$$

$$\underline{\underline{\frac{1}{G_2} = \frac{1}{4\pi\varkappa_1}\left(\frac{1}{R_2}-\frac{1}{R_3}\right)}}$$

b) $$P = G_1 (T_1 - T_2)$$
$$= G_2 (T_2 - T_3)$$

$$\left.\begin{aligned} T_1 - T_2 &= \frac{P}{G_1} \\ T_2 - T_3 &= \frac{P}{G_2} \end{aligned}\right\} \Rightarrow T_1 - T_3 = P\left(\frac{1}{G_1} + \frac{1}{G_2}\right) \equiv \frac{P}{G}$$

$$P = G (T_1 - T_3)$$

$$\underline{\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2}}$$

Fourier's lov for det effektive kuleskallet:

$$j_r = -\varkappa_{eff} \frac{dT}{dr} ; \quad R_1 < r < R_3$$

$$P = 4\pi r^2 j_r = -4\pi \varkappa_{eff} r^2 \frac{dT}{dr}$$

$$dT = -\frac{P}{4\pi \varkappa_{eff}} \frac{dr}{r^2}$$

$$T(r) - T_1 = -\frac{P}{4\pi \varkappa_{eff}} \left(\frac{1}{R_1} - \frac{1}{r}\right) ; \quad R_1 < r < R_3$$

$r \to R_3^-$:

$$T_3 - T_1 = -\frac{P}{4\pi \varkappa_{eff}}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

$$P = G\,(T_1 - T_3)$$

$$\frac{1}{G} = \frac{1}{4\pi \varkappa_{eff}}\left(\frac{1}{R_1} - \frac{1}{R_3}\right) = \frac{1}{G_1} + \frac{1}{G_2}$$

$\Rightarrow$

$$\frac{1}{\varkappa_{eff}}\left(\frac{1}{R_1} - \frac{1}{R_3}\right) = \frac{1}{\varkappa_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{1}{\varkappa_2}\left(\frac{1}{R_2} - \frac{1}{R_3}\right)$$

$$\frac{1}{\varkappa_{eff}}\frac{1}{R_1}\left(1 - \frac{1}{4}\right) = \frac{1}{\varkappa_1}\frac{1}{R_1}\left(1 - \frac{1}{2}\right) + \frac{1}{\varkappa_2}\frac{1}{2R_1}\left(1 - \frac{1}{2}\right)$$

$$\frac{3}{4}\frac{1}{\varkappa_{eff}} = \frac{1}{2\varkappa_1} + \frac{1}{4\varkappa_2}$$

$$\frac{3}{\varkappa_{eff}} = \frac{2\varkappa_2 + \varkappa_1}{\varkappa_1 \varkappa_2}$$

$$\underline{\varkappa_{eff} = \frac{3\varkappa_1 \varkappa_2}{\varkappa_1 + 2\varkappa_2}}$$

c) Ta feks ligning for $T(r)$ for innerste kuleskall; $r \to R_2^-$

$$T_2 - T_1 = -\frac{P}{G_1}$$

$$T_2 = T_1 - \frac{P}{G_1}$$

Fra ligning for ytterste kuleskall:

$$T_3 - T_2 = -\frac{P}{G_2}$$

$$T_2 = T_3 + \frac{P}{G_2}$$

Men vi har også: $P = G(T_1 - T_3)$

$$\underline{T_2 = T_1 - \frac{G}{G_1}(T_1 - T_3)}$$

$$\underline{T_2 = T_3 + \frac{G}{G_2}(T_1 - T_3)}$$

Her er nå alle størrelser kjente

Vi setter inn i G_1, G_2, G de oppgitte verdier $R_2 = 2R_1$; $R_3 = 4R_1$
$\varkappa_1 = 2\varkappa_2$

$$G_1 = 8\pi \varkappa_1 R_1$$

$$G_2 = 16\pi \varkappa_2 R_1$$

$$\frac{G}{G_1} = \frac{G_2}{G_1+G_2} = \frac{2\varkappa_2}{2\varkappa_2+\varkappa_1} = \frac{1}{2}$$

$$\frac{G}{G_2} = \frac{G_1}{G_1+G_2} = \frac{\varkappa_1}{\varkappa_1+2\varkappa_2} = \frac{1}{2}$$

Fra ligning for T_2:

$$T_2 = T_1 - \frac{1}{2}(T_1 - T_3)$$

$$= 1000\,K - 400\,K = \underline{\underline{600\,K}}$$

Fra ligning for T_3:

$$T_2 = T_3 + \frac{1}{2}(T_1 - T_3)$$

$$= 200\,K + 400\,K = \underline{\underline{600\,K}}$$

Oppgave 2

a) $$Z = \prod_{j=1}^{N} \sum_{i=1}^{2} e^{-\beta \varepsilon_{ji}}$$

$$= \prod_{j=1}^{N} \left(e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right)$$

$$= \left(e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right)^N = \left(2\cosh\beta\varepsilon\right)^N$$

$$U = -\frac{\partial \ln Z}{\partial \beta}$$

$$= -N \frac{\partial \ln(2\cosh\beta\varepsilon)}{\partial \beta}$$

$$= -N\varepsilon \, \frac{\sinh(\beta\varepsilon)}{\cosh(\beta\varepsilon)} = \underline{\underline{-N\varepsilon \tanh(\beta\varepsilon)}}$$

b) $$\beta\varepsilon \ll 1 \Rightarrow \tanh(\beta\varepsilon) \simeq \beta\varepsilon$$

$$U = -\frac{N\varepsilon^2}{k_B T}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \underline{\underline{\frac{N\varepsilon^2}{k_B T^2}}}$$

c) $T\,dS = C_V\,dT$

Reversibel T-utjevning, varmeisolert system.

$$\Delta S = \Delta S_1 + \Delta S_2 = 0$$

1. lov:

$$0 = \Delta U + W_{max} \Rightarrow W_{max} = -\Delta U$$

$$\Delta U = \Delta U_1 + \Delta U_2$$

$$\Delta S_1 = \int_{T_1}^{T_S} \frac{C_V}{T}\,dT = \frac{N\varepsilon^2}{2k_B}\left(\frac{1}{T_1^2} - \frac{1}{T_S^2}\right) < 0$$

$$\Delta S_2 = \int_{T_2}^{T_S} \frac{C_V}{T}\,dT = \frac{N\varepsilon^2}{2k_B}\left(\frac{1}{T_2^2} - \frac{1}{T_S^2}\right) > 0$$

$$\Delta U_1 = \int_{T_1}^{T_S} C_V\,dT = \frac{N\varepsilon^2}{k_B}\left(\frac{1}{T_1} - \frac{1}{T_S}\right) < 0$$

$$\Delta U_2 = \int_{T_2}^{T_S} C_V\,dT = \frac{N\varepsilon^2}{k_B}\left(\frac{1}{T_2} - \frac{1}{T_S}\right) > 0$$

T_S bestemmes fra $\Delta S = 0$

$$\frac{N\varepsilon^2}{2k_B}\left(\frac{1}{T_1^2} + \frac{1}{T_2^2} - \frac{2}{T_S^2}\right) = 0$$

$$T_s^2 = \frac{2T_1^2 T_2^2}{(T_1^2 + T_2^2)}$$

$$T_s = \frac{\sqrt{2}\, T_1 T_2}{\sqrt{T_1^2 + T_2^2}}$$

$$W_{max} = -(\Delta U_1 + \Delta U_2)$$

$$= -\frac{N\varepsilon^2}{k_B}\left(\frac{1}{T_1} + \frac{1}{T_2} - \frac{2}{T_s}\right)$$

$$= -\frac{N\varepsilon^2}{k_B T_1 T_2}\left[T_1 + T_2 - \sqrt{2}\sqrt{T_1^2 + T_2^2}\right]$$

$$= \frac{N\varepsilon^2}{k_B T_1 T_2}\left\{\sqrt{2}\sqrt{T_1^2 + T_2^2} - (T_1 + T_2)\right\}$$

For fullstendighetens skyld viser vi også at dette arbeidet er positivt

$$f(x) \equiv \frac{2(T_1^2 + T_2^2)}{(T_1 + T_2)^2} = \frac{2(1+x^2)}{(1+x)^2} \quad ; 0 < x \leq 1$$

$$x = \frac{T_2}{T_1} \qquad f(x) = 2 \ ; \ x = 0$$

$$f(x) \geq 1 \ ; 0 \leq x \leq 1 \qquad f(x) = 1 \ ; \ x = 1$$

$$W_{max} = \frac{N\varepsilon^2}{k_B}\left(\frac{T_1+T_2}{T_1T_2}\right)\left[\sqrt{f(x)} - 1\right]$$

$f(x) \geq 1$; $0 \leq x \leq 1$

$\Longrightarrow$

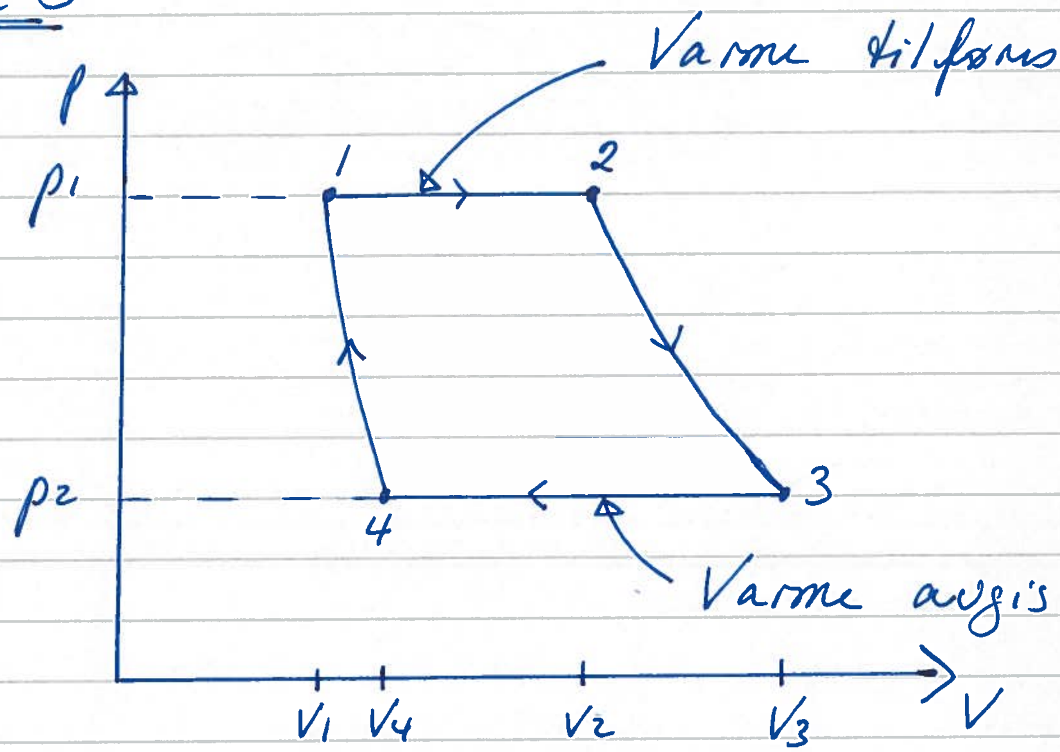
$W_{max} \geq 0$

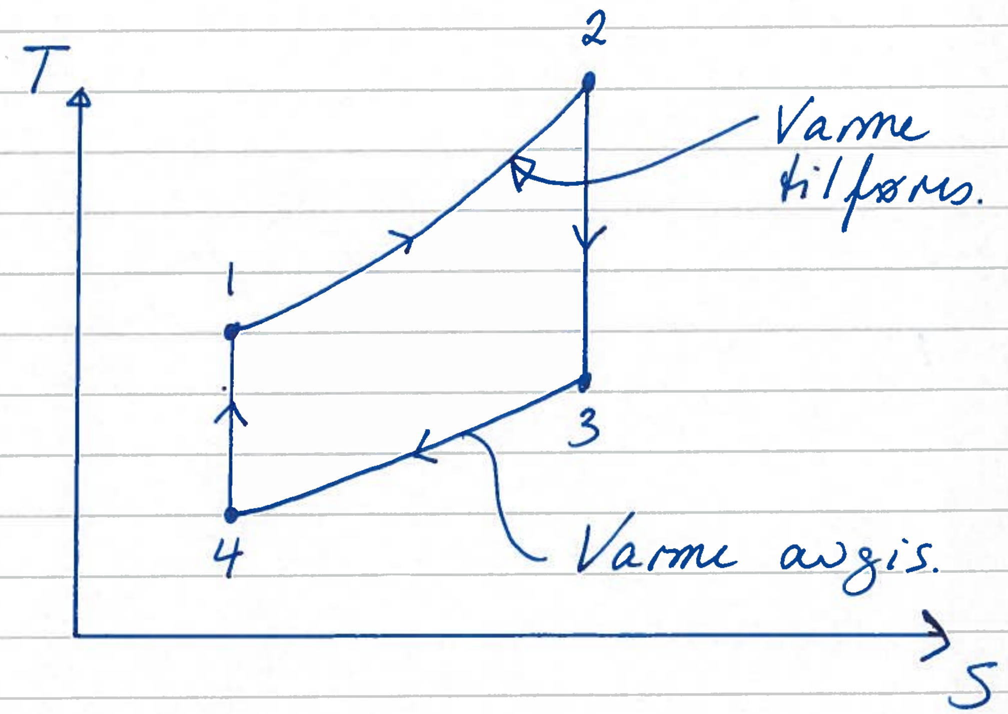
$W_{max} = 0$ når $f(x) = 1$.

Det skjer når $x = 1$ ɔ: $T_1 = T_2$

Oppgave 3

a)





b) Tilført varme : $1 \rightarrow 2$

Avgitt varme : $3 \rightarrow 4$

TDI : $T\,dS = C_p\,dT$

$$Q_t = C_p (T_2 - T_1) > 0$$

$$\underline{Q_a = C_p (T_4 - T_3) < 0}$$

$$W = Q_t + Q_a$$
$$= C_p \left(T_2 - T_1 + T_4 - T_3\right)$$

$$\eta = \frac{W}{Q_t} = 1 - \frac{|Q_a|}{Q_t}$$

$$= 1 - \left(\frac{T_3 - T_4}{T_2 - T_1}\right)$$

Adiabatligning: Vi ønsker å eliminere T-er til fordel for p-er og bruker derfor

$$T p^{\frac{1-\gamma}{\gamma}} = \text{const}$$

langs adiabatene 2→3 og 4→1.

2→3: $$T_2 p_1^{\frac{1-\gamma}{\gamma}} = T_3 p_2^{\frac{1-\gamma}{\gamma}} \quad (*)$$

4→1: $$T_4 p_2^{\frac{1-\gamma}{\gamma}} = T_1 p_1^{\frac{1-\gamma}{\gamma}} \quad (**)$$

$$\eta = 1 - \frac{T_4}{T_1} \frac{\left(\frac{T_3}{T_4} - 1\right)}{\left(\frac{T_2}{T_1} - 1\right)}$$

Fra ⊗⊗ finner vi:

$$\frac{T_4}{T_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1-\gamma}{\gamma}} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Ved å dividere ⊗ med ⊗⊗ finner vi:

$$\frac{T_3}{T_4} = \frac{T_2}{T_1} \Rightarrow \frac{\frac{T_3}{T_4} - 1}{\frac{T_2}{T_1} - 1} = 1$$

$$\eta = 1 - \frac{T_4}{T_1} = \underline{\underline{1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}}}$$

$\frac{\gamma-1}{\gamma}$ øker monotont med γ

$\frac{p_2}{p_1} < 1 \Rightarrow \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$ minker monotont

med $\gamma \Rightarrow \eta$ øker monotont med γ.

$C_p^{(1)} = \frac{5}{2} N k_B$ $\qquad$ $C_p^{(2)} = \frac{7}{2} N k_B$

$\gamma^{(1)} = 5/3$; $\gamma^{(2)} = 7/5$

$$\gamma^{(1)} > \gamma^{(2)} \Rightarrow$$

η er større for ēn-atomig idell gass enn to-atomig