Eksamen TFY4165

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Løsningsforslag

Oppgave 1

a)

$$Z = Z_1^N$$

$$\underline{\underline{Z_1 = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3}}}$$

$$\varepsilon_1 = 0, \quad \varepsilon_2 = \varepsilon, \quad \varepsilon_3 = -\varepsilon$$

$$Z_1 = 1 + e^{-\beta\varepsilon} + e^{\beta\varepsilon}$$

$$= \underline{\underline{1 + 2\cosh(\beta\varepsilon)}}$$

b)

$$\langle\varepsilon\rangle = -\frac{\partial \ln Z_1}{\partial\beta} = -\frac{\partial}{\partial\beta}\ln(1+2\cosh\beta\varepsilon)$$

$$= -\frac{1}{1+2\cosh\beta\varepsilon}\cdot 2\varepsilon\sinh(\beta\varepsilon)$$

$$= \underline{\underline{-2\varepsilon\,\frac{\sinh\beta\varepsilon}{1+2\cosh\beta\varepsilon}}}$$

$\beta \to 0$: $\quad \cosh(\beta\varepsilon) \simeq 1$; $\sinh\beta\varepsilon \simeq \beta\varepsilon$

$$\underline{\underline{\langle\varepsilon\rangle \simeq -\frac{2}{3}\varepsilon\beta^2 \to 0}}$$

$\beta \to \infty$: $\quad \cosh(\beta\varepsilon) \gg 1 = 1$

$$\langle\varepsilon\rangle \simeq -2\varepsilon\,\frac{\sinh\beta\varepsilon}{2\cosh\beta\varepsilon} \simeq -\varepsilon\tanh\beta\varepsilon$$

$$\simeq \underline{\underline{-\varepsilon}}$$

c)

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

Per partikkel:

$$c_V = \left(\frac{\partial \langle \varepsilon \rangle}{\partial T}\right)_V$$

$$= -k_B \beta^2 \frac{\partial \langle \varepsilon \rangle}{\partial \beta}$$

$$= k_B \beta^2 2\varepsilon \left\{ \frac{\varepsilon \cosh(\beta\varepsilon)}{1+2\cosh\beta\varepsilon} - \frac{2\varepsilon \sinh(\beta\varepsilon)\cosh(\beta\varepsilon)}{(1+2\cosh\beta\varepsilon)^2} \right\}$$

$$= 2k_B (\beta\varepsilon)^2 \frac{(2+\cosh(\beta\varepsilon))}{(1+2\cosh\beta\varepsilon)^2}$$

Låg temperatur-grense : $\beta\varepsilon \to \infty$

$$C_V \simeq k_B (\beta\varepsilon)^2 e^{-\beta\varepsilon} \to 0$$

Alle partiklane er samla i lågaste energi-nivå. Her er det ingen energi-fluktuasjonar, grunntilstanden har ikkje evne til å lagre varme.

Svaret er i samsvar med ekv. partisjon-prinsippet, då systemet ikkje har nokon kvadratiske fridomsgrader.

Oppgåve 2

a : A
b : D
c : C
d : A
e : D
f : D
g : C
h : C

Oppgave 3

a) 2D fartsfordeling

$$F_{2D}(v) = 2\pi v\, dv \;\frac{m}{2\pi k_B T}\, e^{-\frac{m}{2}\vec{v}^2}$$

$$\vec{v}^2 = v_x^2 + v_y^2$$

$$\langle v \rangle = \int_0^\infty dv\; v\, F_{2D}(v)$$

$$= 2\alpha \underbrace{\int_0^\infty dv\; v^2\, e^{-\alpha v^2}}_{\equiv I(\alpha)} \quad ; \quad \alpha \equiv \frac{m}{2k_BT}$$

$$= 2\alpha\, I(\alpha)$$

$$I(\alpha) = -\frac{\partial}{\partial\alpha} \int_0^\infty dv\; e^{-\alpha v^2} = -\frac{\partial}{\partial\alpha}\left(\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right)$$

$$= \frac{1}{4}\,\frac{\sqrt{\pi}}{\alpha^{3/2}}$$

$$\langle v \rangle = 2\alpha\;\frac{1}{4}\sqrt{\frac{\pi}{\alpha}}\;\frac{1}{\alpha} = \sqrt{\frac{\pi}{4\alpha}}$$

$$= \underline{\underline{\sqrt{\frac{\pi}{2}}\sqrt{\frac{k_BT}{m}}}}$$

$$\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle$$

$$= \int_0^\infty dv\; v^2\, F_{2D}(v)$$

$$= 2\alpha \int_0^\infty dv\; v^3\, e^{-\alpha v^2} \quad ; \quad \alpha \equiv \frac{m}{2k_BT}$$

$$= \alpha \int_0^\infty dx \; x \; e^{-\alpha x}$$

$$= \alpha \left(-\frac{d}{d\alpha}\right) \int_0^\infty dx \; e^{-\alpha x}$$

$$= \alpha \left(-\frac{d}{d\alpha}\right) \frac{1}{\alpha} = \frac{1}{\alpha}$$

$$\underline{\underline{\langle v^2 \rangle = 2 \frac{k_B T}{m}}}$$

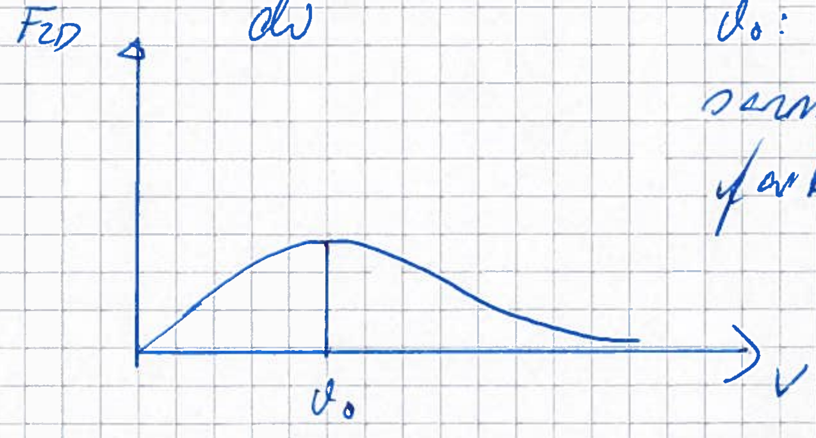
$$\underline{\underline{\langle v^2 \rangle - \langle v \rangle^2 = \langle (v - \langle v \rangle)^2 \rangle \geq 0}}$$

b) Fartsfordelinga:

$$F_{2D}(v)\, dv = \frac{m}{2\pi k_B T} \, 2\pi v \, dv \; e^{-\frac{m}{2 k_B T} v^2}$$

Den mest sannsynlege farten finnast ved å sjå for kva v $F_{2D}(v)$ er størst. Det finn vi frå likninga

$$\frac{dF_{2D}}{dv} = 0$$



v_0: Mest sannsynlege farten.

$$\frac{dF_{2D}}{dv} = K\left(e^{-\alpha v^2} - 2\alpha v^2 e^{-\alpha v^2}\right)$$

$$= 0 \quad \Rightarrow$$

$$1 - 2\alpha v_0^2 = 0$$

$$v_0 = \sqrt{\frac{1}{2\alpha}} = \underline{\underline{\sqrt{\frac{k_B T}{m}}}}$$

$$v_0 = \sqrt{\frac{1.38 \cdot 10^{-23} \cdot 300}{5.314 \cdot 10^{-26}}}$$

$$= \underline{\underline{279\ m/s}}$$

c)

$$\langle v_x^2 \rangle = \sqrt{\frac{m}{2\pi k_B T}} \int_{-\infty}^{\infty} dv_x \, v_x^2 \, e^{-\frac{m}{2k_B T} v_x^2}$$

$$= \sqrt{\frac{m}{2\pi k_B T}} \frac{1}{2} \sqrt{\pi} \frac{2k_B T}{m} \sqrt{\frac{2k_B T}{m}}$$

$$= \frac{k_B T}{m} \Rightarrow \underline{\underline{\langle E_{kin} \rangle = \frac{1}{2} k_B T}}$$

$$\langle v_x \rangle = 0$$

$$\underline{\underline{\langle p_x \rangle = 0}}$$