NTNU
Norwegian University of Science and Technology
Department of Physics

Associate Professor Morten Kildemo, Phone: 73593211; Mobile: 93287744

TFY4195 Optikk (Optics – basic course)

Examination, May, 2008, Time: 09.00 – 13.00

Allowed aid
Level C: "Specified printed and handwritten references allowed. Simple electronic calculator (scientific)." Mathematical reference books are allowed, such as "BETA Mathematics Handbook" (Råde; Westergren) or "Matematisk Formelsamling" (Rottmann) or "Fysikaliske Størrelser" (Øgrim). No lap-top computer, electronic notebook, or similar, is allowed.

Evaluation/grades
Total number of points of the written examination is 100. These will constitute the basis for evaluation. The following table recommended by NTNU will be used for converting to A, B, C, …-scale.

A: 100-90 points
B: 89-80 points
C: 79-60 points
D: 59-50 points
E: 49-40 points
F: 39-0 points

Section A: Geometric Optics

A1. Geometric Optics –fundamental concepts

a) State Fermat's original principle. [3p]

b) How may perfect imaging of a point source be accomplished using lenses or mirrors in line with Fermat's principle (no detailed calculations required). [3p]

c) What are the main approximations in Gaussian, first order optics [3p].

d) Given a slab of transparent material (in air) with a gradient refractive index n(x). A typical ray path is shown in Figure A1.3.

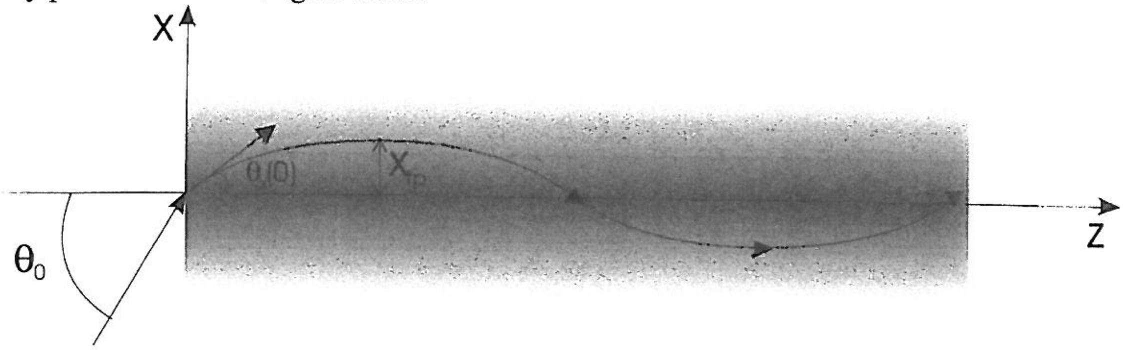


Figure A1.3. The gradient index slab. A ray is incident from the left at (x=0,z=0) and is incident in the x-z plane, (x and z are in arbitrary units).

$$n(x)=\begin{cases}1.3-|x| & for\,|x|<0.3\\ 1 & for\,|x|>0.3\end{cases}$$

Given the ray equation (or eikonal equation)

$$\nabla n(x)=\frac{d}{d\sigma}(n\bar{s})$$

where $\bar{s}=\frac{d\bar{r}}{d\sigma}$, σ is the geometric path length and $\bar{r}=(x,y,z)$.

Write the ray equation in component form. Find an expression for $\theta_z(x)$ (see Figure A1.3) as a generalization of Snell's law, for a given $\theta_z(x=0)$. [3p]

Why does the ray turn at sufficiently small θ_0? Find the x-value (x_{tp}) for the turning point (see Figure A1.3), given $\theta_0=30°$. [3p]

At what angle of incidence, θ_0 (see Figure A1.3), will the ray cease to be bound (i.e. exit from the side of slab). [3p]

[Total of 18 p]

A2. Geometric Optics –analysis of optical systems

A simple telescope is given by two lenses (objective and eyepiece(ocular)) separated by a distance $L=f_o+f_e$, where f_o and f_e are the focal lengths of the objective and the eyepiece (ocular), respectively.

a) Calculate the system transfer matrix for two thin lenses separated by the distance $L=f_e+f_o$, and hence find the total refractive power. [5p]

A large astronomical telescope is investigated having $\mathbf{f_o=16m}$ and $\mathbf{f_e=4\ cm}$. The diameter of the objective is **D=80 cm**. The objective is known to be the aperture stop in the system, while the eyepiece (or ocular) is known to be the field stop of the system (diameter D_{FS} unknown).

b) What is the angular magnification of the system. (relative to you looking at a distant star with only your eye). [3p]

c) What is the definition of the Aperture stop, the Entrance and Exit pupils, and the Field stop. Find the location and the size of the exit pupil (ExP). Where should your eye be located in order to maximize the luminosity, and how does it here fit well to your eye. [12p] 2p (14p)

In order to reduce chromatic and spherical aberrations the objective and the eyepiece consists both of compound lenses. The Figure A2.1 shows the objective and the eyepiece (ocular) for such a system. The focal points and the principal points are indicated in the Figure A2.1.

d) Sketch the rays through the system, coming from a distant star (you may use the attached answer sheet). In particular, sketch two parallel rays of which one is passing trough the nodal point. Sketch also a ray coming in parallel to the optic axis. [5p] (6p)

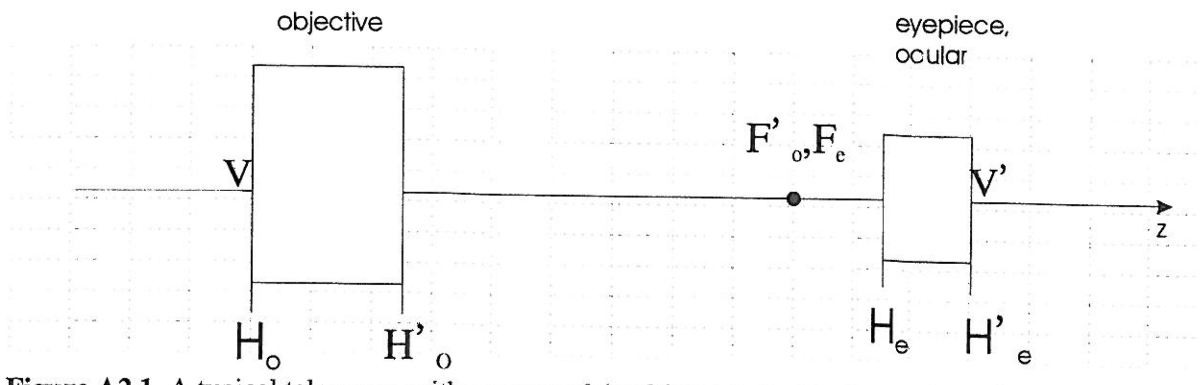


Figure A2.1. A typical telescope, with compound (multielement) objective and eyepiece. H_O -first principal plane of objective. H'_O -second principal plane of objective. H_e -first principal plane of eyepiece. H'_e -second principal plane of eyepiece. F'_o-back focal point of objective. F_e-front focal point of the eyepiece (ocular).

[Total of 25 points]

28 points + 3 points

Section B: Wave Optics

B1. Polarization

A plane electromagnetic wave, with a given polarization can be written as:

$$\vec{E} = \left(E_{01}\hat{e}_1 e^{ikz} + E_{o2}\hat{e}_2 e^{i(kz+\delta)}\right)e^{-i\omega t} \qquad \text{(B1.1)}$$

where $\hat{e}_1, \hat{e}_2$ are othogonal unit vectors.

2p

a) Write the electric field in equation (B1.1) as a Jones vector. Describe the polarization state (polarisation ellipse) of the following normalized Jones vectors (located at specific point in space): [5p]

1p 1p 1p

$$\begin{bmatrix}1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

3p

b) What are the limitations of the Jones formalism and what are the advantages of the Stokes vector formalism. [3p]

c) You are analyzing a system with unpolarized light which is sent trough a polarizer with the transmission axis oriented at 45 degrees with the x_1-axis, followed by a quarterwave plate with the fast axis along the x_1-axis, What is the polarization state (polarisation ellipse) after the retarder, ***with respect to the cartesian laboratory coordinates*** (x_1, x_2)**!**

If you now add at the end a polarizer rotating around the optic axis, what is the resulting intensity recorded on a detector as a function of orientation of this polarizer. [5p]

[Total number of 13 points]

Pol $\begin{bmatrix}1\\1\end{bmatrix}$ 2p

$\delta = \frac{\pi}{2}$ 1p

Left circ.

$E(\alpha)$ 2p

$I =$ 2p

Calc 3p or shortcut

(12p)

total of 17p

+ 5 points

B2 Interference from thin film

Figure B2 shows the recorded Reflectivity (R_s) of SiO_2/c-Si, as a function of angle of incidence, at a fixed wavelength (λ=600 nm). (recorded with a polarizer, a filter and a photodiode and a reference sample). The dispersive refractive index is given by

$$n(\lambda, SiO_2) = 1.448 + \frac{3642}{\lambda^2} \quad \text{, where } \lambda \text{ is in nm.}$$

In order to avoid the effect of dispersion, the reflectivity (R_s) was recorded as a function of the angle of incidence. Two consecutive maxima are estimated at φ_i =38.3° and φ_{i+1} =47.25°.

The thin film reflection formula is given as:

$$r = r_{01} + t_{01}r_{12}t_{10}e^{i\delta} + t_{01}r_{12}r_{10}r_{12}t_{10}e^{i2\delta} + \ldots = \frac{r_{01} + r_{12}e^{i\delta}}{1 + r_{01}r_{12}e^{i\delta}}, \text{ where } \delta = \frac{4\pi d_1}{\lambda}\sqrt{n_1^2 - n_0^{\,2}\sin^2\varphi_o}$$

where the subscripts for polarisation have been omitted.

a) Calculate the thickness of this layer, using the refractive index given above (you may here neglect the slow angular dependence of r_{01} and r_{12}). [5p]

b) From considerations of the thin film reflection formulae, deduce and sketch the reflectivity for the bare c-Si substrate with no film as a function of angle of incidence. [4p]

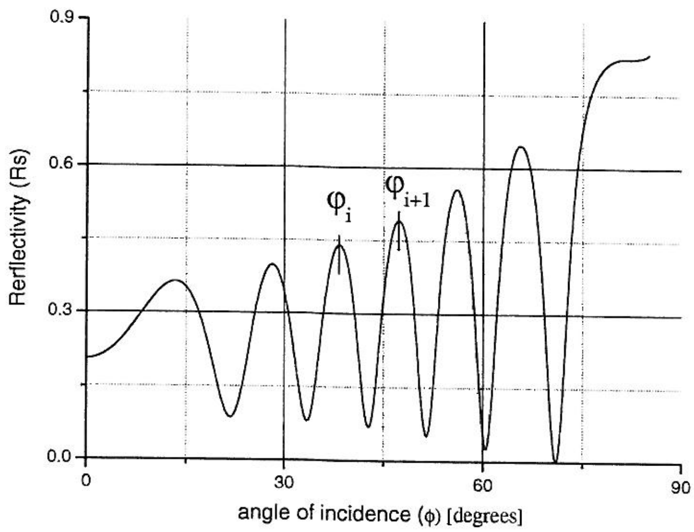


Figure B2. Reflectivity R_s as a function of angle of incidence, (λ=600 nm)

c) Describe and make a sketch of an experiment in order to visualize the fringes (similar to the ones in Figure B2, but now on a screen. (e.g. similar to the the Haidinger fringes, hint: use an incoherent extended source and a lens). [5p]

[Total of 14 points]

B3. Diffraction and Fourier Optics

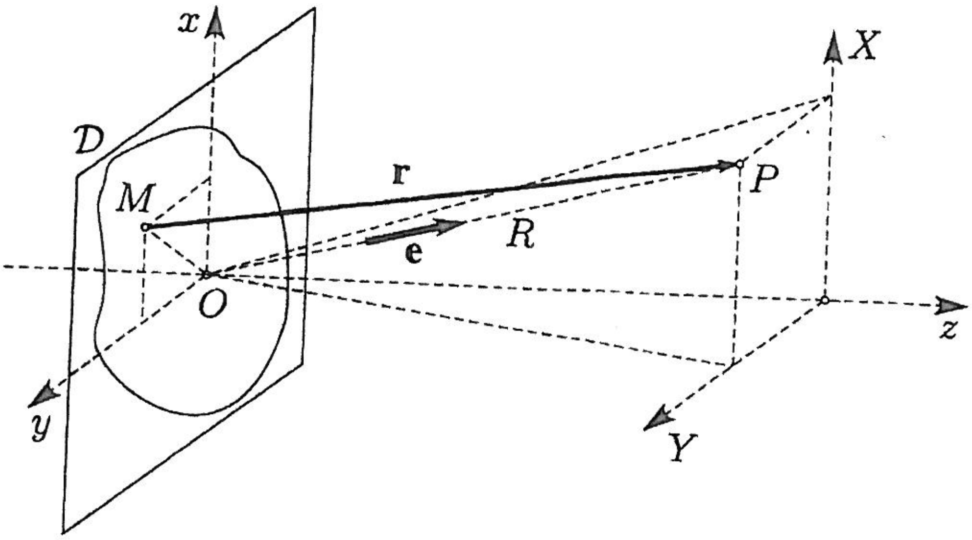


Figure B3.1. Typical example of geometry for the analysis of diffraction from an aperture.

The Huygens Fresnel Principle is here given by the first Rayleigh Sommerfeld solution:

$$U(X,Y,z)=\frac{1}{i\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x,y,0)\frac{\exp(ikr)}{r}\cos(\theta)dxdy \qquad \text{(B3.1)}$$

where

$$r=\sqrt{(X-x)^2+(Y-y)^2+z^2} \qquad \text{(B3.2)}$$

and

$$\cos\theta=\frac{z}{r} \qquad \text{(B3.3)}$$

You may find useful formulas in the appendix.

a) Derive the Fresnel diffraction formulae from the above diffraction formulae. [5p]

b) The phase function of a lens may in the paraxial approximation be derived to be :

$$t_{lens}(x,y)=\exp\left[-\frac{ik}{2f}(x^2+y^2)\right] \qquad \text{(B3.4)}$$

where f is the focal length of the lens.

-show that the lens with the transmission function t_{lens} in equation (B3.4) may be used to obtain the Fourier transform (i.e. the Fraunhofer diffraction condition) of an aperture. You should neglect the aperture of the lens and assume plane wave incident on the aperture from the left. [3p]

-sketch an optical system using a point source (from the left) and two lenses, that will give the Fourier Transform of the aperture transmission function in the observation plane (to the right). Give the "positions" of the lenses with respect to their focal lengths. [3p]

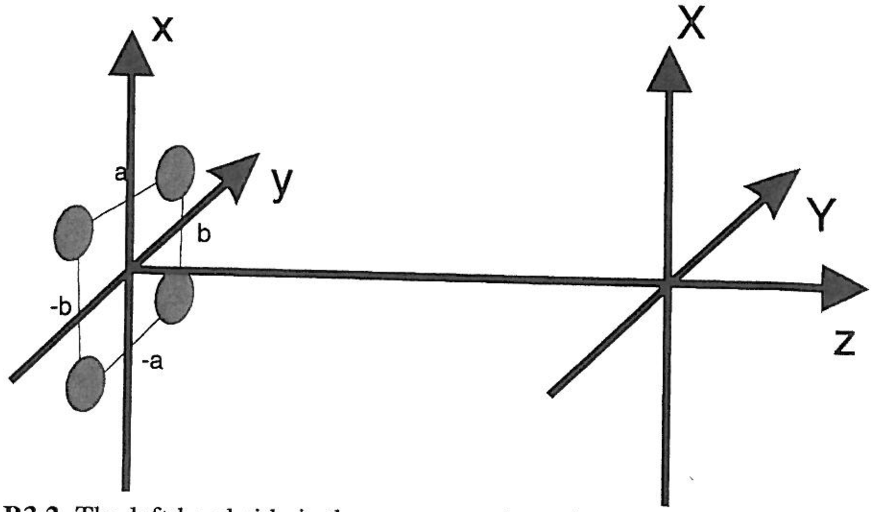


Figure B3.2. The left hand side is the aperture, where the grey represents the holes, with diameter D. Plane waves are incoming from the left onto the aperture. We want to observe the Fraunhofer diffraction pattern in the observation plane (right hand side). You should insert the lenses at appropriate positions in order to obtain the Fraunhofer condition.

Your system (or equivalently the Fraunhofer diffraction pattern) of the aperture in Figure B3.3 is to be calculated and analyzed:

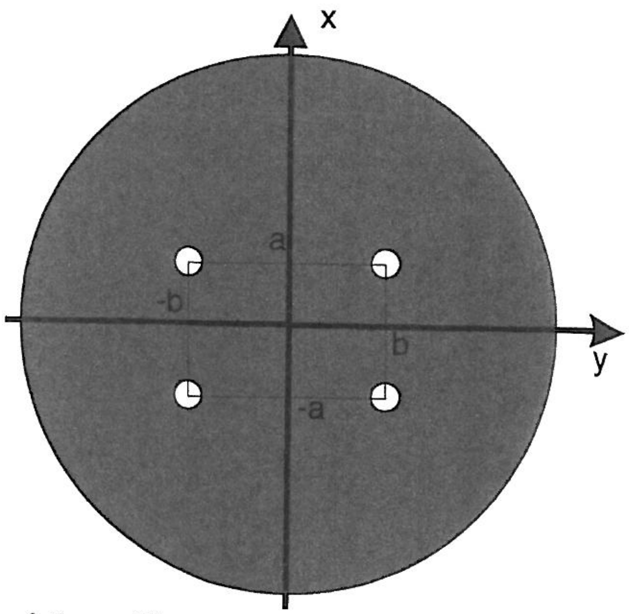


Figure B3.3. Aperture of the problem to be analyzed. The aperture is opaque and considered as infinitely large, while each hole has a diameter D.

c) Define the amplitude transmission function, and show that the transmission function of the aperture in Figure B3.3 may be written in the form: [4p]

$$t(x,y) = \delta(x-a, y+b) \otimes circ(\frac{\rho}{D/2}) + \ldots$$

where the common circ function is defined in the appendix.

FT 2p

d) Calculate analytically the

-scalar electric field

-observable intensity

in the observation plane in the Fraunhofer condition assuming a plane incident wave. [6p]

e) A real diffraction pattern is shown in Figure B3.4. Find from your calculations an expression for the observation plane coordinates (X,Y) at which one should observe maxima in the intensity. [~~5p~~] (3p) -2p

- what happens if the circular holes become infinitely small. [2p]
- what happens if the wavelength $\lambda \rightarrow 0$. [2p]

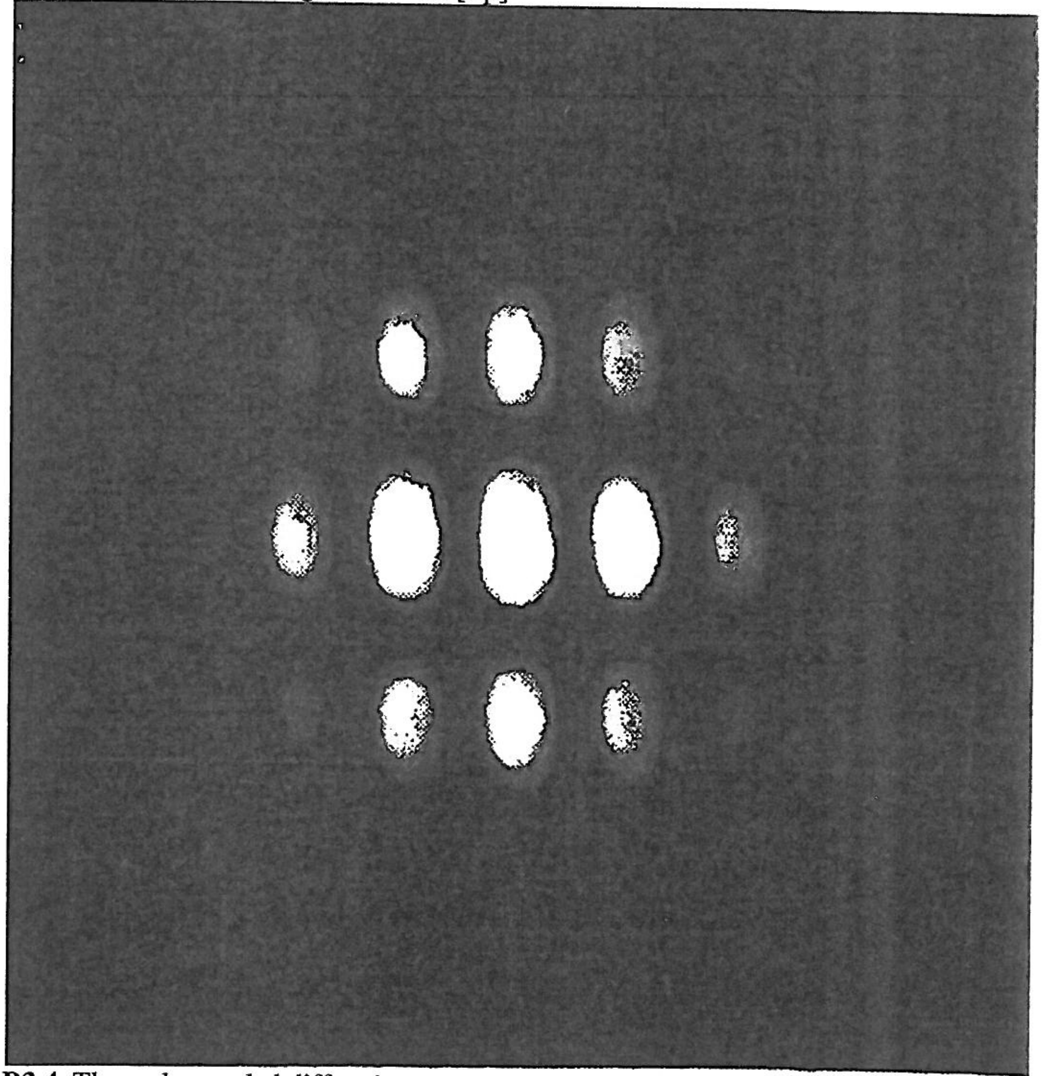


Figure B3.4. The real recorded diffraction pattern from the aperture being studied. The figure does not show all diffraction orders predicted by the basic theory.

[Total of 30 points]

Appendix

The 2D Fourier Transform of a general aperture transmission function is given by :

$$T(k_X, k_Y) = \mathfrak{F}\{t(x, y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t(x, y)\exp[-i(k_X x + k_Y y)]dxdy$$

The Fraunhofer diffraction integral is given by :

$$U(X, Y, z) = \frac{1}{i\lambda z}e^{ikz}e^{ik\frac{(X^2+Y^2)}{2z}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, y, 0)\exp[-i(k_X x + k_Y y)]dxdy$$

where

$$k_X = \frac{kX}{z},\ k_Y = \frac{kY}{z},\ k = \frac{2\pi}{\lambda}$$

or alternatively (depending on how the approximations are performed)

$$U(X, Y, z) = \frac{1}{i\lambda R}e^{ikR}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, y, 0)\exp[-i(k_X x + k_Y y)]dxdy$$

where

$$k_X = k\sin\theta_X = \frac{kX}{R} \approx \frac{kX}{z},\ k_Y = k\sin\theta_Y = \frac{kY}{R} \approx \frac{kY}{z},\ R = \sqrt{X^2 + Y^2 + z^2}$$

Some useful functions in optics, Fourier Transforms and its properties

The circ function is defined by:

$$circ(\frac{\rho}{b}) = \begin{cases} 1 & for\ |\rho| < b \\ 0 & elsewise \end{cases}$$

The rect function is defined by :

check

sinc( )

$$rect(\frac{x}{w}) = \begin{cases} 1 & for\ |x| < \frac{w}{2} \\ 0 & elsewise \end{cases}$$

The dirac delta function

$$\delta(x) = \begin{cases} \infty & for\ x = 0 \\ 0 & elsewise \end{cases}$$

$$\int_{-\infty}^{\infty}\delta(x)dx = 1$$

The Fourier Transform of the circ function with radius d:

$$T(\kappa) = \mathfrak{F}\left\{circ(\frac{\rho}{d})\right\} = \pi d^2 \, 2\frac{J_1(\kappa d)}{\kappa d},$$

where $\rho = \sqrt{x^2 + y^2}$, and $\kappa = \sqrt{k_X{}^2 + k_Y{}^2}$,
and J_1 *is the Bessel function of first Kind.*

Table B3.1. Properties of the J_1 *function.*

B	0	1.22	1.63	2.33	2.68	3.33
$2\frac{J_1(\pi B)}{\pi B}$	1	0	0.017	0	0.004	0

The Fourier Transform of the rect function

$$\mathfrak{F}\left\{rect(\frac{x}{w})\right\} = w\frac{\text{sinc}(xw/2)}{xw/2},$$

Properties of the sinc function :
Define

$$\text{sinc}(x) = \frac{\sin(x)}{x},$$

Then $\text{sinc}(x) = 1$ *for* $x=0$,

$|\text{sinc}(x)|$ - *maximum for* $x=(2m+1)\pi/2$, $m=0,1,2,\ldots$,

$\text{sinc}(x) \to 0$ *for* $x=m\pi$, $m=0,1,2,\ldots$,

The Fourier Transform of the Dirac delta function

$$\mathfrak{F}\{\delta(x)\} = 1$$

The Fourier transform of a comb function

$$\mathfrak{F}\left\{\sum_{n=-N}^{n=N} \delta(x - na)\right\} = \frac{\sin\left((2N-1)k_X a/2\right)}{\sin\left(k_X a/2\right)}$$

The shift property

$$\mathfrak{F}\{f(x - x_0)\} = \exp(ik_X x_0)\,\mathfrak{F}\{f(x)\}$$

The convolution theorem :

The convolution is defined by : $g(x) = \int_{-\infty}^{\infty} f(\xi)h(x-\xi)d\xi = f(x) \otimes h(x)$

then by the convolution theorem $\mathfrak{F}\{g(x)\} = \mathfrak{F}\{f(x)\}\mathfrak{F}\{h(x)\}$

The sifting property of the Dirac delta function :

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(x-a)$$

The Linearity + scaling property

$$\mathfrak{F}\{\alpha g(x)+\beta f(x)\} = \alpha\mathfrak{F}\{g(x)\}+\beta\mathfrak{F}\{f(x)\}$$

Fresnel reflection and transmission coefficients:

$$r_\perp \equiv r_s = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}; \quad t_\perp \equiv t_s = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_\parallel \equiv r_p = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}; \quad t_\parallel \equiv t_p = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$

Boundary conditions for reflection/transmission at interface:

$$B_{2z} - B_{1z} = 0 \tag{1.5.2}$$

$$D_{2z}-D_{1z}=\varepsilon_2 E_{2z}-\varepsilon_1 E_{1z}=0 \tag{1.5.3}$$

$$\mathbf{E}_{2t} - \mathbf{E}_{1t} = 0 \tag{1.5.4}$$

$$\mathbf{H}_{2t} - \mathbf{H}_{1t} = (\mathbf{B}_{2t} - \mathbf{B}_{1t})/\mu_0 = 0 \tag{1.5.5}$$

Maxwell's equations:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}=0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = 0$$

Material equations

$$\mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{B} = \mu\mathbf{H}$$

Jones matrices for common components

Linear Polarizer:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

General waveplate:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$$

Rotation matrix :

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

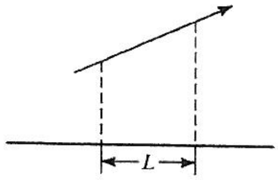
Jones Matrix T rotated by α:

$$R(-\alpha)TR(\alpha)$$

TABLE 18-1 SUMMARY OF SOME SIMPLE RAY-TRANSFER MATRICES

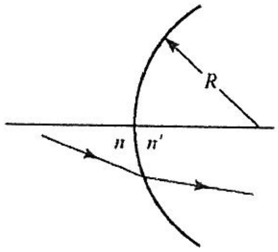
Translation matrix:

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \mathfrak{T}$$



Refraction matrix, spherical interface:

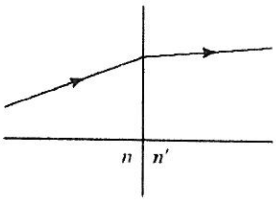
$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n-n'}{Rn'} & \dfrac{n}{n'} \end{bmatrix}$$



$(+R)$: convex
$(-R)$: concave

Refraction matrix, plane interface:

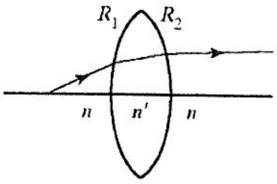
$$M = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n}{n'} \end{bmatrix}$$



Thin-lens matrix:

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

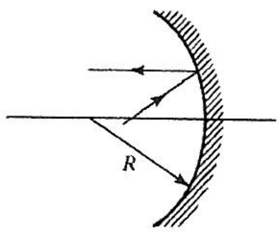
$$\frac{1}{f} = \frac{n' - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$



$(+f)$: convex
$(-f)$: concave

Spherical mirror matrix:

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$$



$(+R)$: convex
$(-R)$: concave

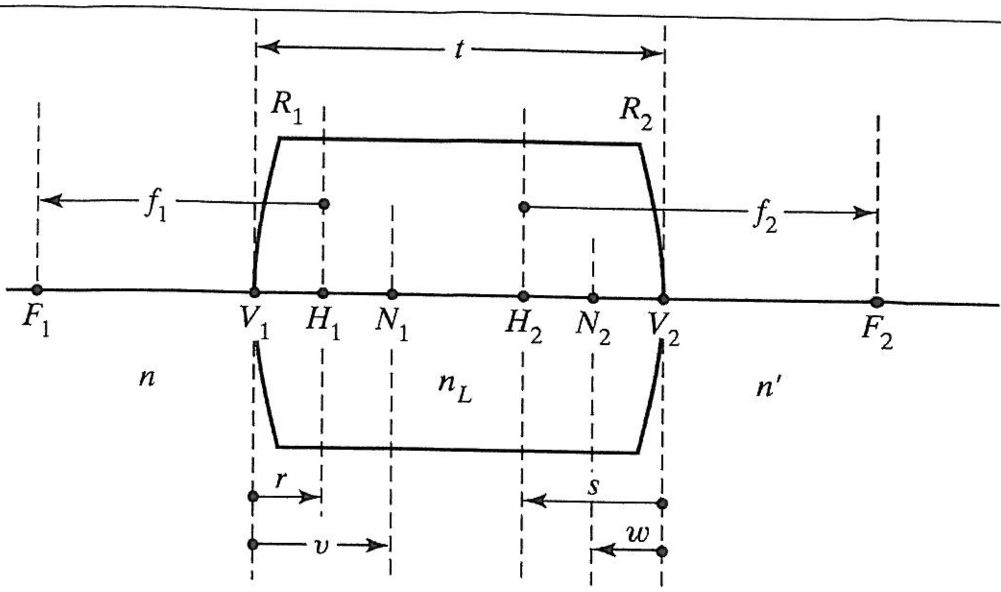


TABLE 18-2 CARDINAL POINT LOCATIONS IN TERMS OF SYSTEM MATRIX ELEMENTS

$p = \frac{D}{C}$	F_1	Located relative to input (1) and output (2) reference planes
$q = -\frac{A}{C}$	F_2	
$r = \frac{D - n_0/n_f}{C}$	H_1	
$s = \frac{1 - A}{C}$	H_2	
$v = \frac{D - 1}{C}$	N_1	
$w = \frac{n_0/n_f - A}{C}$	N_2	
$f_1 = p - r = \frac{n_o/n_f}{C}$	F_1	Located relative to principal planes
$f_s = q - s = -\frac{1}{C}$	F_2	

TABLE 2-1 SUMMARY OF GAUSSIAN MIRROR AND LENS FORMULAS

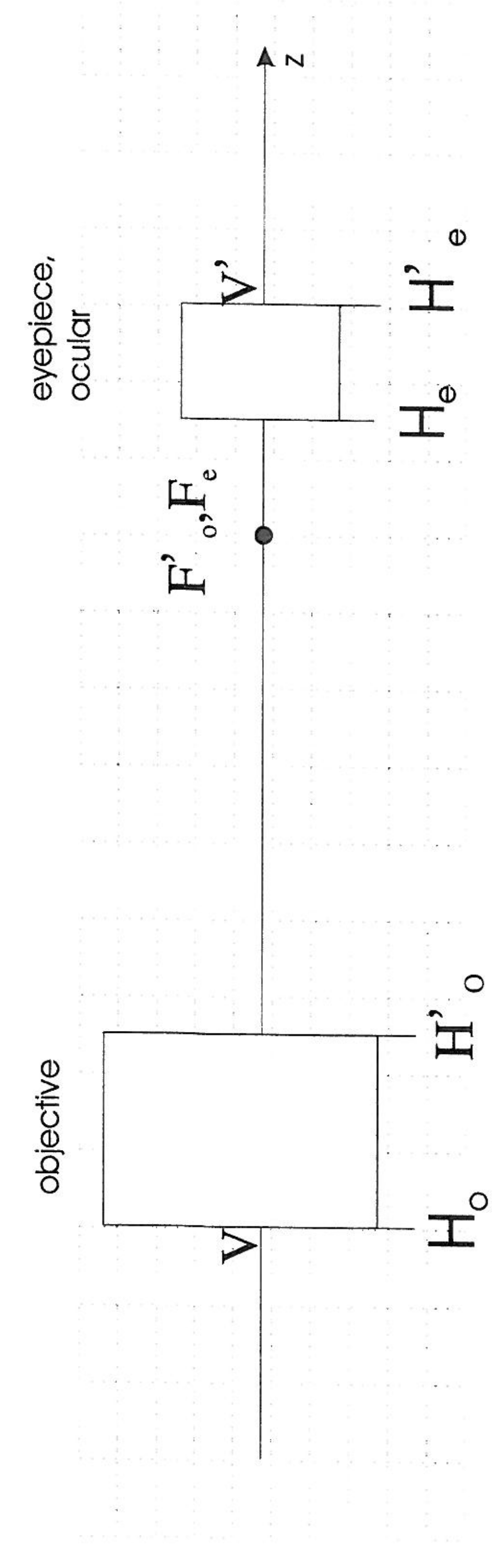
	Spherical surface	Plane surface
Reflection	$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}, f=-\frac{R}{2}$ $m=-\frac{s'}{s}$ Concave: $f>0, R<0$ Convex : $f<0, R>0$	$s'=-s$ $m=+1$
Refraction Single surface	$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{R}$ $m=-\frac{n_1 s'}{n_2 s}$ Concave: $R<0$ Convex : $R>0$	$s'=-\frac{n_2}{n_1}s$ $m=+1$
Refraction Thin lens	$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$ $\frac{1}{f}=\frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ $m=-\frac{s'}{s}$ Concave: $f<0$ Convex : $f>0$	

Newtonian imaging equation :

$$ff'=zz'$$

where z,z' refers to focal points.

Suggested answer sheet, problem A2.2



Suggested answer sheet, problem B2.b