NTNU
Norwegian University of Science and Technology
Department of Physics

Professor Morten Kildemo, Mobile: 93287744
Faglig kontakt under eksamen: Lars M. Sandvik Aas, Mobile: 99013637

TFY4195 Optikk (Optics – basic course)

Examination, May 25th, 2012, Time: 09.00 – 13.00

Allowed aid

Level C:
Typegodkjent kalkulator, med tomt minne, i samsvar med NTNUs regler. Trykte hjelpemidler: “Matematisk Formelsamling” (Rottmann), ”Størrelser og Enheter i Fysikk og Teknikk,” (O. Øgrim og B. E. Lian) eller ”Fysiske Størrelser og Enheter,” (C. Angell og B. E. Lian).

Simple electronic calculator (scientific) with an empty memory, in accordance with the NTNU rules. No lap-top computer, electronic notebook, or similar, is allowed. Mathematical reference books are allowed, such as “BETA Mathematics Handbook” (Råde; Westergren) or “Matematisk Formelsamling” (Rottmann), ”Størrelser og Enheter i Fysikk og Teknikk,” (O. Øgrim og B. E. Lian) or ”Fysiske Størrelser og Enheter,” (C. Angell og B. E. Lian).

Evaluation/grades

Total number of points of the written examination is 100. These will constitute the basis for evaluation. The following table recommended by NTNU will be used for converting to A, B, C, …-scale.

A: 100-90 points
B: 89-80 points
C: 79-60 points
D: 59-50 points
E: 49-40 points
F: 39-0 points

Section A: Geometric Optics [50p]

A.1. [Total 7p]

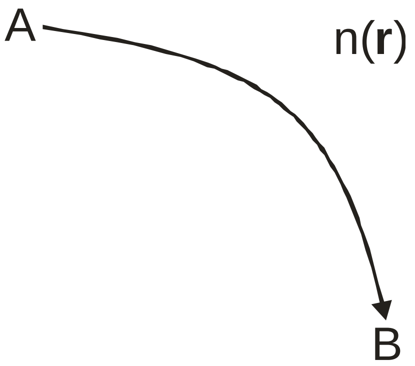


Figure A.1. Geometric ray in inhomogeneous medium given by refractive index n(**r**).

Imagine that a geometric ray curves in an isotropic inhomogeneous transparent medium, as shown in Figure A.1. i) By inspecting the curvature of the ray, what may you qualitatively state about the refractive index profile n(**r**) and briefly discuss which path the ray would take between points A and B (no calculation required). ii) Present a short description of an observable phenomenon in nature where such ray curving takes place.

You are given the following equations as reference:

$$\nabla n = \frac{d\ \ n\hat{s}}{d\sigma}\ ,\ S(P_0, P_1) = \int_{P_0}^{P_1} n(\sigma) d\sigma$$

A.2. [Total 19 points]

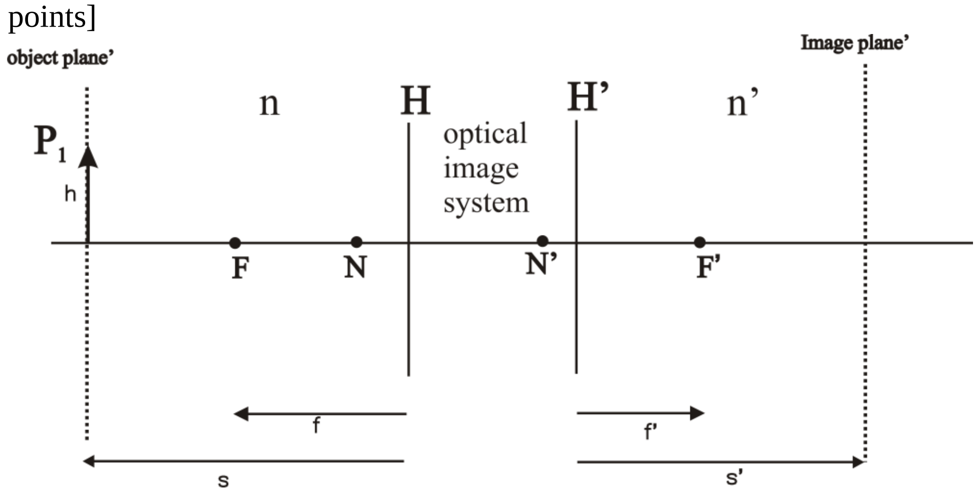


Figure A.2. A general imaging system represented by its cardinal points.

The figure shows a general optical system represented by its cardinal points. The paraxial transfer matrix between the principal planes (hovedplan) H and H' are given by:

$$\mathbf{M}_{HH'} = \begin{bmatrix} 1 & 0 \\ C & \frac{n}{n'} \end{bmatrix}$$

a) [7p] i) What is the transverse magnification between the planes H and H', and what is the angular magnification for rays into H and out of H'. ii) Show that $C = -\frac{1}{f'}$, (where f' is the back-focal length), by considering am incoming ray to H that is parallel to the optic axis.

b) [4p] For the general imaging system in Figure A.2, perform graphical imaging from an object point P_1 to an image point P_1'.

c) [8p] Find the total paraxial transfer matrix from the object plane to the image plane,with respect to the principal points (i.e. using s and s'), see Figure A.2. From this matrix, derive the Gaussian imaging equation with respect to cardinal points:

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n'}{f'},$$

$$M_t = \frac{h'}{h} = -\frac{n}{n'}\frac{s'}{s}$$

A.3. [Total 23 points].

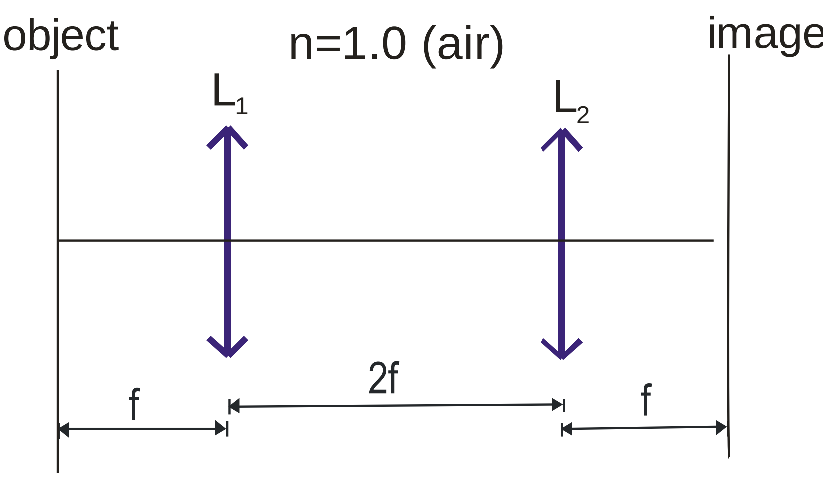


Figure A.3.1. Standard 4f imaging system with object and image in air. L_1 and L_2 are two identical lenses with focal length f.

a)[6p] Using paraxial transfer matrices, show that a 4f imaging system, see Figure A.3.1, (with both object and image in air), has transverse magnification M_t=-1, angular magnification M_α=-1, and is an afocal imaging system.

A rectangular glass bowl is filled with index matching liquid. Hence the glass and the liquid has refractive index n'=1.5. Air-bubbles are flowing into the bowl from the bottom of the container, and we want to make and image and estimate the size of the bubbles, using a CCD camera with a 1x1 cm^2 CCD chip (A_3=1 cm). Illumination is assured by a separate system. The lenses have focal lengths f=175 mm, and are effectively limited by circular apertures of diameter 6 cm. The distance from the bubbles to the container wall is f=175 mm, while the distances between the lenses L_1 and L_2 is 2f=350 mm, and from L_2 to the CCD chip the distance is also f=175 mm.

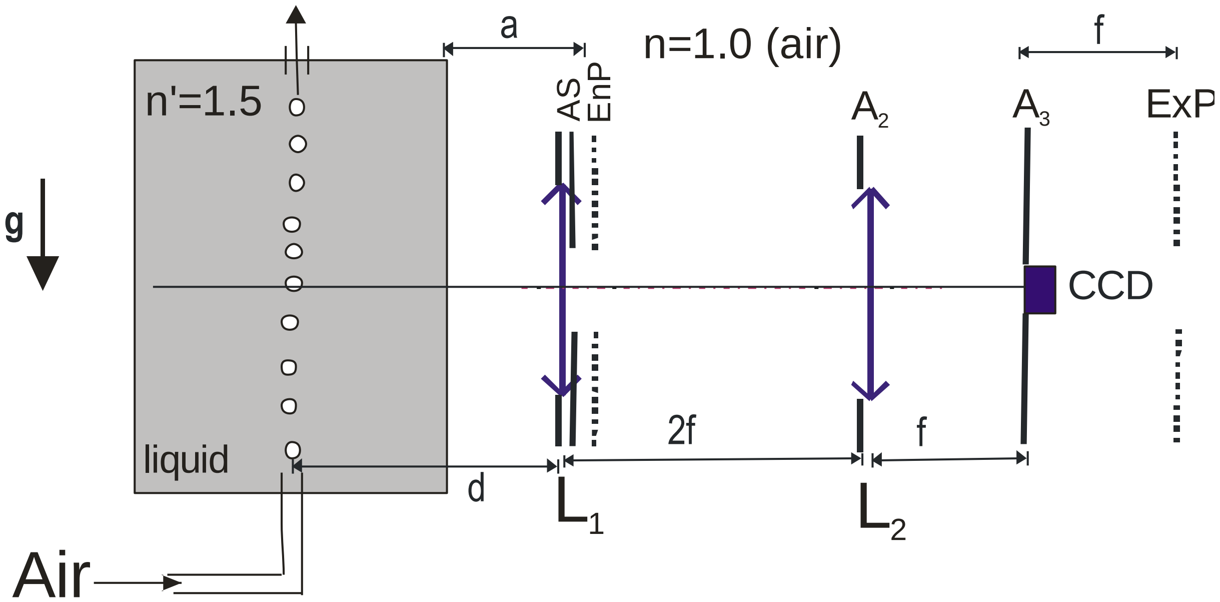


Figure A.3.2 Air bubbles are floating up through a rectangular glass container filled with a liquid index-matched to the glass. The bubbles are to be imaged onto the CCD chip, using basically the 4f imaging system. The focal lengths are f=175 mm, the container is 2f=350 mm wide and the bubbles are assumed to float midway through the container. The AS, and the EnP and the ExP of the system is shown in the drawing. Note that the figure is not drawn to scale.

b) [6p] Find the distance d, in order to form a paraxial image of the bubbles onto the CCD chip, using the standard 4f system from part (a). Find the transverse magnification (relative size of image compared to the object).

The Aperture Stop (AS), is a circular aperture, that is inserted immediately after the lens L_1, with diameter AS=4 cm. The Entrance Pupil (EnP) is located $a = f/2 = 87.5[mm]$ from the container, and the Exit pupil (ExP) of the imaging system is located a distance f from the CCD aperture, as shown in the drawing in Figure A3.2. The diameter of the EnP is D_{EnP}=4 cm, and the diameter of the ExP is D_{ExP}=4 cm.

c) [5p] Find the Field Stop (FS), the Exit Window (ExW), the Entrance Window (EnW) and the angular field of view of the system in Figure A3.2. Give both the size and the position of the EnW.

d) [4p] Assuming two ideal lenses L_1 and L_2 and assuming that the current AS in Figure A.3.2 is removed, explain what aberrations you would expect from the plane interface from liquid to air. Make a sketch to demonstrate how such aberrations affect the final image?

e) [4p] If an aperture of small opening-diameter is located midway between the two lenses (L_1 and L_2), where will the EnP and the ExP be located. Discuss the advantages and disadvantages of the new system, where a descriptive sketch should be included.

Section B Wave-Optics [50p]:

B.1 **Polarisation** [Total 15p]

a) [6p] The Jones vector for a general polarization state can be represented by:

$$\bar{J} = \begin{bmatrix} 1 \\ \chi \end{bmatrix}$$

Sketch (without proof) the polarisation ellipse for the following Jones vectors: $\chi = 1, \pm i, 1+i$.
For $\chi = 1$ and $\chi = i$, write the corresponding electric field propagating in the z-direction ($\hat{\mathbf{k}} = \hat{\mathbf{z}}$) in the form:

$$E(x,y,z,t) = E_x(z,t)\hat{x} + E_y\ z,t\ \hat{y}.$$

b) [9p] We hereby analyze the *rotating analyzer ellipsometer*: Light is incident with angle of incidence, (θ) (θ is here typically between 50-75^o) on a plane isotropic surface, with Jones matrix

$$\begin{bmatrix} r_p & 0 \\ 0 & r_s \end{bmatrix},$$

where r_p and r_s are the Fresnel reflection coefficients A polarizer is inserted in the beam before reflection, with transmission axis oriented 45 degrees counter-clockwise rotated from the incidence plane of the sample (looking into the beam). After reflection, a polarizer (so-called analyzer) is rotated an angle (α), with respect to the incidence plane (still looking into the beam), and is finally followed by a detector. You may assume monochromatic light.

i)[6p]-Make a sketch of the system to be analyzed, particularly describing the incidence plane.(Hint. Either a 3D drawing or several projections are needed).
-Find an expression for the Jones vector after the analyzer (as a function of α, r_p and r_s). The Jones vector must be reported with respect to the incidence plane.

ii) [3p] Find the intensity (I_D) on the detector for the analyzer rotated to $\alpha = 45^o$.

B.2 **Interference** [Total 16p]

A sample consists of a double side polished silicon wafer with air on both sides. The sample is illuminated with collimated light (plane waves), at an angle of incidence θ_0, and we record the specular transmitted light.

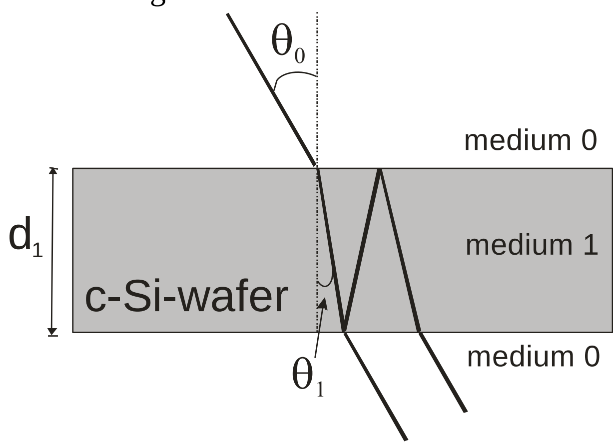


Figure B.2.1. A plane wave is transmitted through a low doped c-Si wafer in air. Only one internal reflection is included in the analysis. θ_0, θ_1 are the angle of incidence and the refracted angle, respectively.

a) [6p] Let us only consider two internal reflections, see Figure B2.1. Write down an expression for the transmission coefficients t_p and t_s, and calculate the transmittance for s-polarized light (T_s), as a function of the Fresnel coefficients r_{10}, t_{01} and t_{10}. You may presume the phase shift from the layer (c-Si wafer) to be given by:

$$\frac{2\pi}{\lambda} n_1 \cos\theta_1 d_1$$

b) [6p] Make an estimate of the thickness of the wafer by analyzing the interference fringes from the recorded transmittance using s-polarized light as a function of wavelength, with angle of incidence θ_0=30^o, see $T_s(\lambda)$ in Figure B2.2. You may assume the refractive index for silicon in this wavelength range is approximately $n \approx 3.42$.

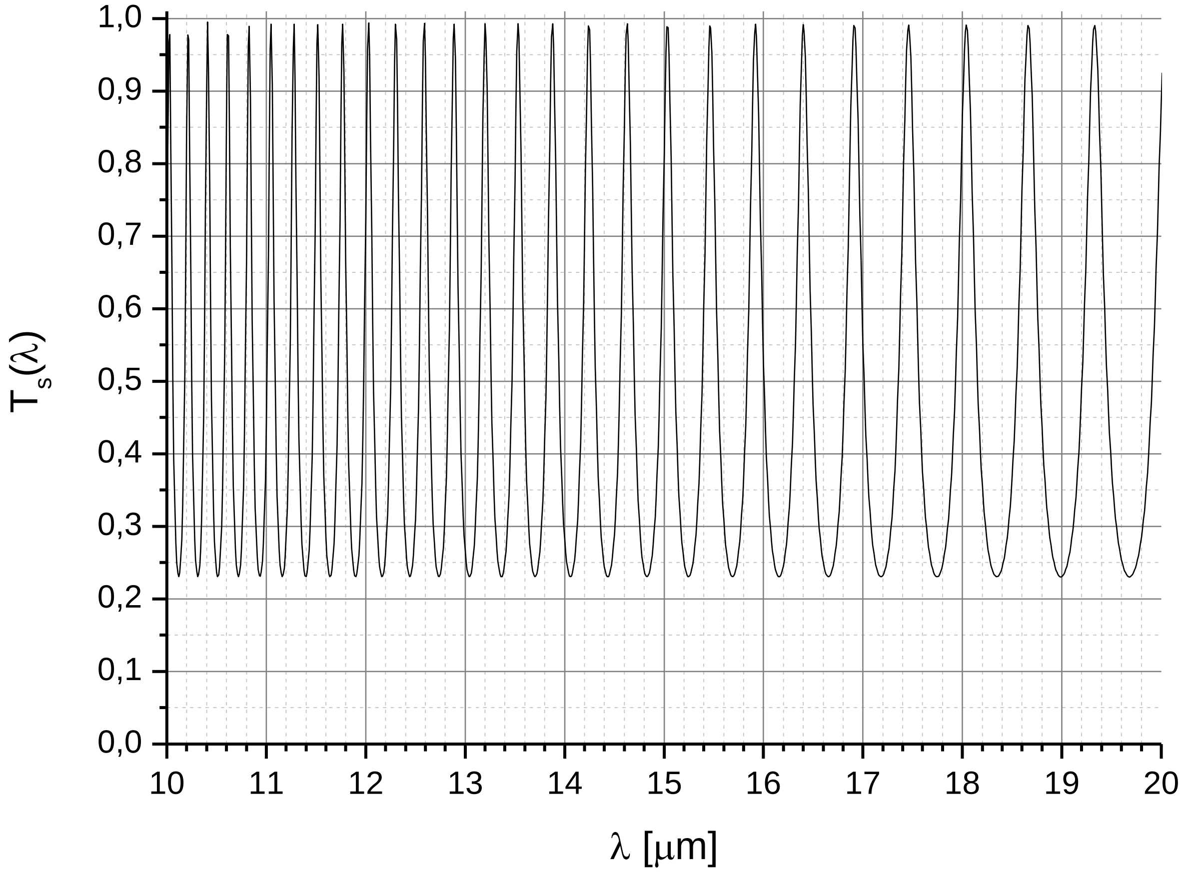


Figure B2.2. The transmittance as a function of wavelength (simulated using data for typical un-doped crystalline silicon). The angle of incidence is θ_0=30^o.

c)[4p] Show from your results in part (a) why the transmittance in Figure B2.2 oscillates to approximately 1 (You may alternatively explain it qualitatively). Determine from the figure the maximum and minimum reflectivity.

B3.1. Diffraction, coherence and Fourier optics [Total. 19p]

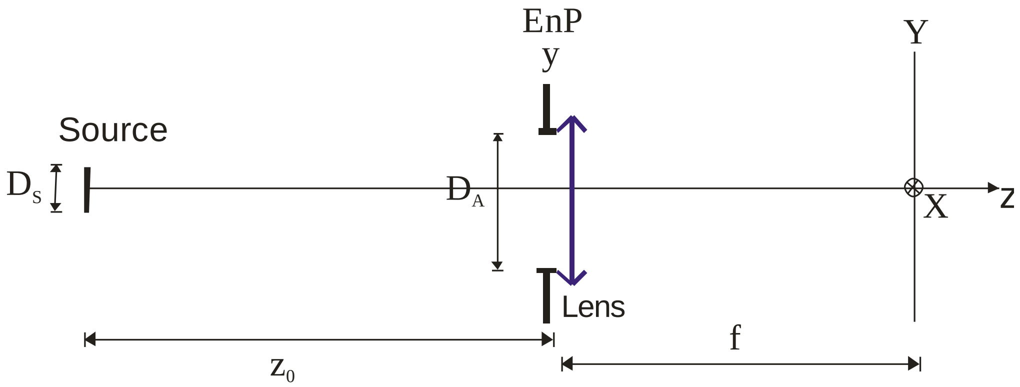


Figure B.3.1. A source with diameter (D_S) illuminates the EnP of the image system consisting here of a single lens. We assume that light from the source is filtered and thus monochromatic.

A source, (such as a distant star) is illuminating the Entrance Pupil (EnP) of an imaging system consisting of a single lens, as seen in Figure B.3.1.

a) [8p] Assuming first that the source as a monochromatic distant point source (i.e. you may assume spatially coherent monochromatic plane wave illumination), and given a circular EnP of diameter D_A, find an expression for the electric field (complex notation) and the intensity observed on the observation screen (or a CCD camera). Justify the validity of the method you use. The transmission function for a lens is given by

$$t_{lens}(x,y) = e^{i\frac{2\pi}{\lambda}n\Delta_0} e^{-i\frac{2\pi}{\lambda}\frac{x^2+y^2}{2f}} .$$

b) [5p] If the source is extended and has a diameter D_S, it has been shown that the transverse coherence length is given by

$$l_t = \frac{1.22\lambda z_0}{D_s}$$

Briefly discuss how this result can be obtained, and estimate how far away the source must be in order to observe diffraction effects from the EnP?

c) [6p] To characterize our system, we use monochromatic (λ=632 nm) collimated laser light (plane wave), and a high resolution CCD camera. Figure B.3.2 shows the recorded diffraction pattern, corresponding to the circular aperture, similar to Figure B.3.1. You are given that the focal length of the lens is f=20 cm. By reading off the Figure B.3.2, estimate the diameter of the aperture.

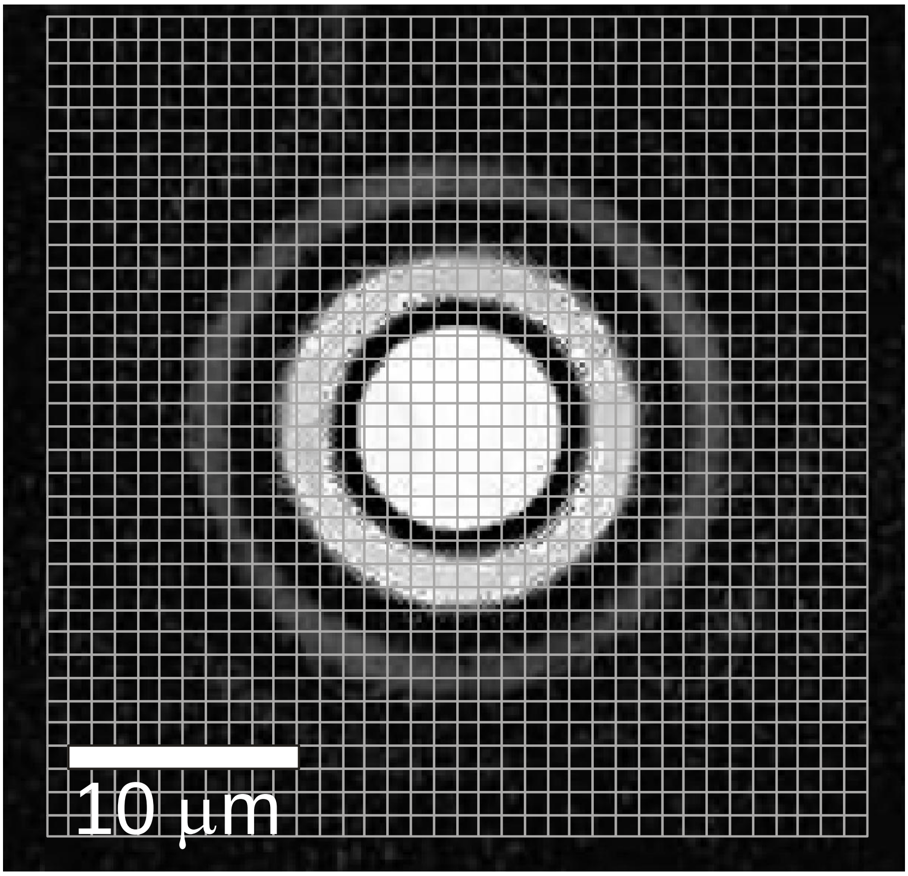


Figure B.3.2. The intensity recorded on the CCD camera. A grid has been added which shows 1μm per grid box.

END OF EXAM

Appendix

The First Rayleigh-Sommerfeld solution:

$$U(X,Y,z)=\frac{1}{i\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x,y,0)\frac{\exp\ ikr}{r}\cos(\theta)dxdy$$

The Fresnel Diffraction integral is given by :

$$U(X,Y,z)=\frac{1}{i\lambda z}e^{ikz}e^{ik\frac{X^2+Y^2}{2z}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x,y,0)\exp\left[\frac{ik}{2z}\ \ x^2+y^2\ \ -i\ \ k_X x+k_Y y\ \right]dxdy$$

The 2D Fourier Transform of a general aperture transmission function is given by :

$$T(k_X,k_Y)=\mathfrak{F}\ \ t(x,y)\ \ =\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t(x,y)\exp\left[-i\ \ k_X x+k_Y y\ \right]dxdy$$

The Fraunhofer diffraction integral is given by :

$$U(X,Y,z)=\frac{1}{i\lambda z}e^{ikz}e^{ik\frac{X^2+Y^2}{2z}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x,y,0)\exp\left[-i\ \ k_X x+k_Y y\ \right]dxdy$$

where

$$k_X=\frac{kX}{z},\ k_Y=\frac{kY}{z}\ ,\ k=\frac{2\pi}{\lambda}$$

or alternatively (depending on how the approximations are performed)

$$U(X,Y,z)=\frac{1}{i\lambda R}e^{ikR}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x,y,0)\exp\left[-i\ \ k_X x+k_Y y\ \right]dxdy$$

where

$$k_X=k\sin\theta_X=\frac{kX}{R}\approx\frac{kX}{z},\ k_Y=k\sin\theta_Y=\frac{kY}{R}\approx\frac{kY}{z}\quad ,\ R=\sqrt{X^2+Y^2+z^2}$$

Some useful functions in optics, Fourier Transforms and its properties

The circ function is defined by:

$$circ(\frac{\rho}{b})\ =\begin{cases}1 & for\ |\rho|<b\\ 0 & elsewise\end{cases}$$

The rect function is defined by :

$$rect(\frac{x}{w})\ =\begin{cases}1 & for\ |\mathrm{x}|<\frac{w}{2}\\ 0 & elsewise\end{cases}$$

The dirac delta function

$$\delta(x) = \begin{cases} \infty & \textit{for } x = 0 \\ 0 & \textit{elsewise} \end{cases} \qquad \textit{(heuristic definition)}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad \textit{(rigorous definition)}$$

The Fourier Transform of the circ function with radius d:

$$T(\kappa) = \mathfrak{F}\left\{circ(\frac{\rho}{d})\right\} = \pi d^2 \, 2\frac{J_1(\kappa d)}{\kappa d},$$

where $\rho = \sqrt{x^2 + y^2}$, and $\kappa = \sqrt{k_X{}^2 + k_Y{}^2}$,
and J_1 *is the Bessel function of first Kind.*

Table Appendix.1. *Properties of the "Besinc" function* $\left(2\frac{J_1(\pi B)}{\pi B}\right)$.

B	*0*	*1.22*	*1.63*	*2.33*	*2.68*	*3.33*
$2\frac{J_1(\pi B)}{\pi B}$	*1*	*0*	*0.017*	*0*	*0.004*	*0*

The Fourier Transform of the rect function

$$\mathfrak{F}\left\{rect(\frac{x}{w})\right\} = w\frac{\sin\ k_X w/2}{k_X w/2},$$

Properties of the sinc function :
Define

$$\mathrm{sinc}(x) = \frac{\sin\ x}{x},$$

Then $\mathrm{sinc}(x) = 1$ *for x=0,*

$|\mathrm{sinc}(x)|$ *- maximum for x=(2m+1)π/2, m=0,1,2,...*

$\mathrm{sinc}(x) \to 0$ *for x=mπ, m=1,2,3,...*

The Fourier Transform of the Dirac delta function

$$\mathfrak{F}\ \delta(x) = 1$$

The Fourier transform of a comb function, (sum of 2N+1 delta functions):

$$\mathfrak{F}\left\{\sum_{n=-N}^{n=N} \delta\ x - na\right\} = \frac{\sin\ 2N+1)k_X a/2}{\sin\ k_X a/2}$$

The function $\frac{\sin^2\ 2N+1\ k_X a/2}{\sin^2\ k_X a/2} \to\ 2N+1\ ^2$, *for* $k_X a/2 = m\pi$, *where* $m = 0, \pm 1, \pm 2, ...$

The shift property

$$\mathfrak{F}\ \ f(x - x_0) = \exp(-ik_X x_0)\,\mathfrak{F}\ \ f(x)$$

The convolution theorem :

The convolution is defined by : $g(x)=\int_{-\infty}^{\infty} f(\xi)h(x-\xi)d\xi \;=f(x)\otimes h(x)$

then by the convolution theorem $\mathfrak{F}\;\; g(x) \;\;=\;\; \mathfrak{F}\;\; f(x)\;\; \mathfrak{F}\;\; h(x)$

The sifting property of the Dirac delta function :

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx \;=f(a)$$

The Linearity + scaling property

$$\mathfrak{F}\;\; \alpha g(x)+\beta f(x) \;\;=\;\; \alpha\mathfrak{F}\;\; g(x) \;\;+\beta\mathfrak{F}\;\; f(x)$$

Fresnel reflection and transmission coefficients:

$$r_\perp \equiv r_s = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}; \quad t_\perp \equiv t_s = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_\parallel \equiv r_p = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}; \quad t_\parallel \equiv t_p = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$

Boundary conditions for reflection/transmission at interface:

$$B_{2z} - B_{1z} = 0$$

$$D_{2z}-D_{1z}=\varepsilon_2 E_{2z}-\varepsilon_1 E_{1z}=0$$

$$\mathbf{E}_{2t} - \mathbf{E}_{1t} = 0$$

$$\mathbf{H}_{2t} - \mathbf{H}_{1t} = (\mathbf{B}_{2t} - \mathbf{B}_{1t})/\mu_0 = 0$$

Maxwell's equations:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = 0$$

Material equations

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

Polarisation, Jones matrices, definition of polarisation ellipse

Linear Polarizer:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

General waveplate:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$$

Rotation matrix :

$$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Jones Matrix T rotated by α:

$$R(-\alpha)\, T R(\alpha)$$

Jones vector for a polarisation ellipse along its principal axis (x',y') in the Figure Pol. ellipse :

$$\bar{J}_{x',y'} = \begin{bmatrix} \cos\varepsilon \\ i\sin\varepsilon \end{bmatrix} = \cos\varepsilon \begin{bmatrix} 1 \\ i\tan\varepsilon \end{bmatrix}, \textit{ where } \tan\varepsilon = \frac{E_{0y'}}{E_{0x'}}$$

The tilted polarisation ellipse Jones vector along the laboratory coordinates is given by:

$$\bar{J}_{x,y} = R(-\alpha)\, \bar{J}_{x',y'}$$

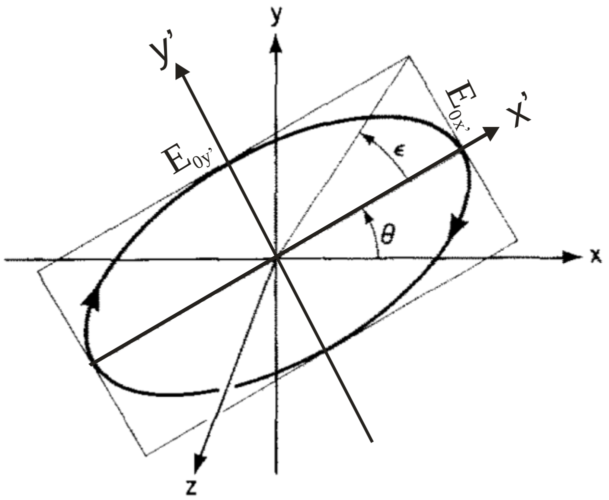
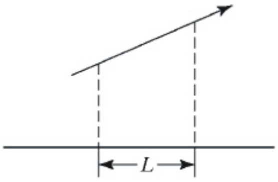


Figure Polarisation ellipse.

TABLE 18-1 SUMMARY OF SOME SIMPLE RAY-TRANSFER MATRICES

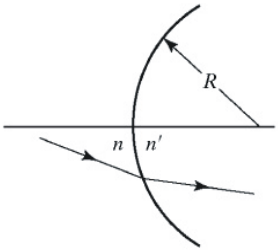
Translation matrix:

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \mathfrak{T}$$



Refraction matrix,
spherical interface:

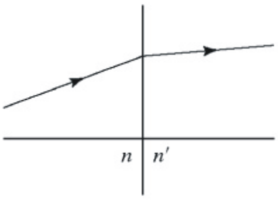
$$M = \begin{bmatrix} 1 & 0 \\ \frac{n-n'}{Rn'} & \frac{n}{n'} \end{bmatrix}$$



$(+R)$: convex
$(-R)$: concave

Refraction matrix,
plane interface:

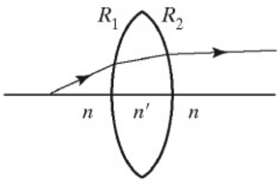
$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix}$$



Thin-lens matrix:

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

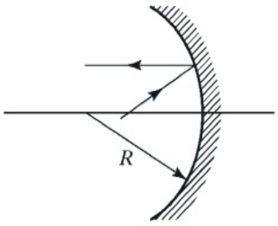
$$\frac{1}{f} = \frac{n'-n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$



$(+f)$: convex
$(-f)$: concave

Spherical mirror
matrix:

$$M = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$



$(+R)$: convex
$(-R)$: concave

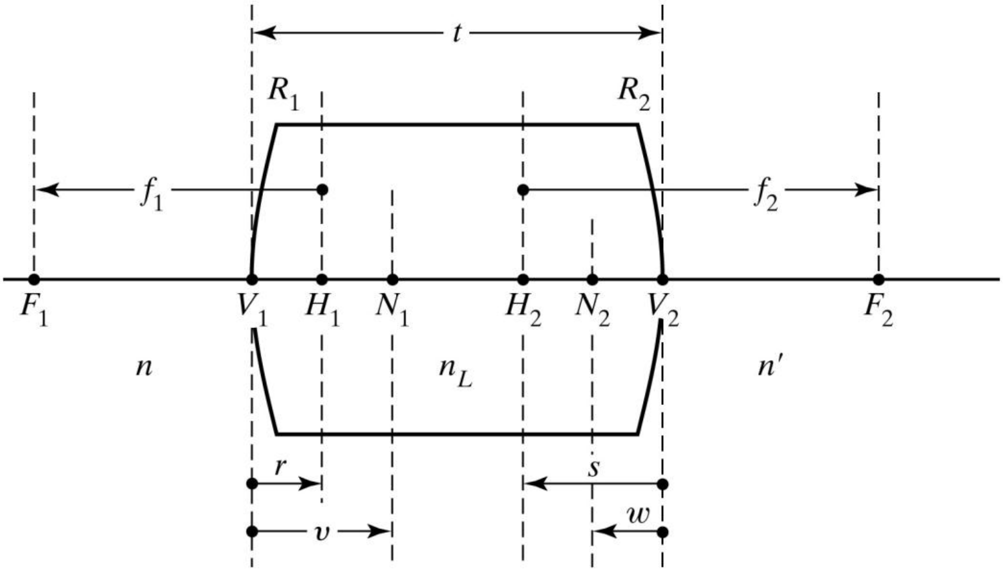


TABLE 18-2 CARDINAL POINT LOCATIONS IN TERMS OF SYSTEM MATRIX ELEMENTS

$p = \frac{D}{C}$	F_1	Located relative to input (1) and output (2) reference planes
$q = -\frac{A}{C}$	F_2	
$r = \frac{D - n_0/n_f}{C}$	H_1	
$s = \frac{1 - A}{C}$	H_2	
$v = \frac{D - 1}{C}$	N_1	
$w = \frac{n_0/n_f - A}{C}$	N_2	
$f_1 = p - r = \frac{n_o/n_f}{C}$	F_1	Located relative to principal planes
$f_s = q - s = -\frac{1}{C}$	F_2	

Recall that in table 18-2, all distances are taken as positive to the right of the plane it refers to, and negative to the left of the plane it refers to.

TABLE 2-1 SUMMARY OF GAUSSIAN MIRROR AND LENS FORMULAS

	Spherical surface	Plane surface
Reflection	$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}, f = -\frac{R}{2}$ $m = -\frac{s'}{s}$ Concave: $f > 0, R < 0$ Convex : $f < 0, R > 0$	$s' = -s$ $m = +1$
Refraction Single surface	$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$ $m = -\frac{n_1 s'}{n_2 s}$ Concave: $R < 0$ Convex : $R > 0$	$s' = -\frac{n_2}{n_1}s$ $m = +1$
Refraction Thin lens	$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$ $\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ $m = -\frac{s'}{s}$ Concave: $f < 0$ Convex : $f > 0$	

Newtonian imaging equation :

$$ff' = zz'$$

where z,z' refers to focal points.

The ray equation is given by:

$\nabla n = \frac{d}{d\sigma}\ n\hat{\mathbf{s}}$, where $\hat{\mathbf{s}} = \frac{d\vec{r}}{d\sigma}$, σ is the geometric path length and $\vec{r} = \ x, y, z$.

Useful differential geometry:

The derivative of the unit tangent vector ($\hat{\mathbf{s}}$) to a curve at a point, can be described by the principal normal to the curve ($\hat{\mathbf{e}}_n$) at that point, and its radius of curvature (R) of the curve at that point:

$$\frac{d\hat{\mathbf{s}}}{d\sigma} = \frac{1}{R}\hat{\mathbf{e}}_n$$

Useful constants:

Planck constant h=6.62617x10^{-34} [J-s].
Speed of light in vacuum c=2.99792x10^{8} [m/s].
Elementary charge e=1.60218x10^{-19} [C].