Problem 1

i) $\begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix}$

2) $\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix}$, $P = \frac{+(n'-n)}{R}$

$P = \frac{1}{f}$, for lens

3) $\begin{pmatrix} \beta & C \\ -P & 1/\beta \end{pmatrix}$

d) must derive

$$\begin{pmatrix} 1 & d'/n' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A' & B \\ C' & D \end{pmatrix}$$

$$= \begin{pmatrix} A + C\frac{d'}{n'} & B + A\frac{d}{n} + D\frac{d'}{n'} + C\frac{dd'}{nn'} \\ C & D + C\frac{d}{n} \end{pmatrix}$$

$\beta = 1$, (Principal planes) hence $A' = D' = 1$

$\therefore$ $A + C\frac{h'}{n'} = 1 \Rightarrow \boxed{h' = \frac{(1-A)}{C} n'}$

$D + C\frac{d}{n} = 1 \Rightarrow \boxed{h = \frac{(1-D)}{C} n}$

also accepting when $n = n' = 1$

(e)

$$\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ +1/f_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f_1 & 1 \end{pmatrix}\begin{pmatrix} 1-\frac{d}{f_1} & d \\ -\frac{1}{f_1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-\frac{d}{f_1} & d \\ \cancel{\frac{1}{f_1}} - \frac{d}{f_1^2} - \cancel{\frac{1}{f_1}} & \frac{1}{f_1}d + 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-\frac{d}{f_1}-\frac{d^2}{f_1^2} & d + \frac{1}{f_1}d^2 + d \\ -\frac{1}{f_2}\left(1-\frac{d}{f_1}\right) - \frac{d}{f_1^2}\left(\frac{-d}{f_2}+1\right) & -\frac{d}{f_2} + \left(-\frac{d}{f_2}+1\right)\left(\frac{d}{f_1}+1\right) \end{pmatrix}$$

$$\begin{pmatrix} 1-\frac{d}{f_1}-\frac{d^2}{f_1^2} & 2d + \frac{d^2}{f_1} \\ -\frac{1}{f_2} + \frac{d}{f_1 f_2} + \frac{d^2}{f_1^2 f_2} - \frac{d}{f_1^2} & 1 - \frac{2d}{f_2} - \frac{d^2}{f_1 f_2} + \frac{d}{f_1} \end{pmatrix}$$

(f)

f) power is $P = -C$

$$\therefore P = -\left[-\frac{1}{f_2} + \frac{d}{f_1 f_2} + \frac{d^2}{f_1^2 f_2} - \frac{d}{f_1^2}\right]$$

Problem 2

i) $U(x,y,z) = e^{i\vec{k}_1\cdot\vec{r}} + e^{i\vec{k}_2\cdot\vec{r}} + e^{i\vec{k}_3\cdot\vec{r}}$

also accept

$$U(x,y,z) = e^{i(\vec{k}_1\cdot\vec{r}+\varphi_1)} + e^{i(\vec{k}_2\cdot\vec{r}+\varphi_2)} + e^{i(\vec{k}_3\cdot\vec{r}+\varphi_3)}$$

where

$$\vec{k}_i\cdot\vec{r} = k_{ix}x + k_{iy}y + k_{iz}z$$

for this case $k_{iy}=0$

and

$$k_{x_1} = k\sin\theta_1$$
$$k_{x_2} = k\sin\theta_2$$
$$k_{x_3} = k\sin\theta_3$$

and

$$k_{z_1} = k\cos\theta_1$$
$$k_{z_2} = k\cos\theta_2$$
$$k_{z_3} = k\cos\theta_3$$

-ote: regarding θ's

(b)

$$U(x,y,0) = e^{ik_{x_1}x} + e^{ik_{x_2}x} + e^{ik_{x_3}x}$$

$$k_{x_1} = k\sin\theta_1$$
$$k_{x_2} = k\sin\theta_2$$
$$k_{x_3} = k\sin\theta_3$$

(c)

$$I = UU^*$$
$$\propto \left[e^{ik_{x_1}x} + e^{ik_{x_2}x} + e^{ik_{x_3}x}\right]\left[\sim\sim\right]^*$$
$$\propto \left[e^{ik_{x_1}x} + e^{ik_{x_2}x} + e^{ik_{x_3}x}\right]\left[e^{-ik_{x_1}x} + e^{-ik_{x_2}x} + e^{-ik_{x_3}x}\right]$$
$$\propto 1 + e^{i(k_{x_1}-k_{x_2})x} + e^{i(k_{x_1}-k_{x_3})x} + e^{i(k_{x_2}-k_{x_1})x} + 1 + e^{i(k_{x_2}-k_{x_3})x} + e^{i(k_{x_3}-k_{x_1})x} + e^{i(k_{x_3}-k_{x_2})} + 1$$

$$= 3 + 2\cos[(k_{x_1}-k_{x_2})x] + 2\cos[(k_{x_1}-k_{x_3})x] + 2\cos[(k_{x_2}-k_{x_3})x]$$

(This for case where arbitrary phases $\varphi_1 = \varphi_2 = \varphi_3 = 0$)

$$k_{x_1} = k\sin\theta_1$$
$$k_{x_2} = k\sin\theta_2$$
$$k_{x_3} = k\sin\theta_3$$

Problem 3

(a)

Fresnel's diffraction integral

$$U(x,y,z) = \frac{1}{i\lambda z} e^{ik[z+\frac{1}{2z}(x^2+y^2)]} \iint_{-\infty}^{\infty} U(x',y',0)\, e^{-i\frac{k}{z}(xx'+yy')} e^{\frac{ik}{2z}(x'^2 \ldots}$$

here $U(x',y',0) = A\,t(x,y)$

also, for Fraunhofer diffraction we have $2z \gg k(x'^2+y'^2)$, hence last term in integral is ... and Fresnel integral reduces to

$$U(x,y,z) = \frac{1}{i\lambda z} e^{ik[z+\frac{1}{2z}(x^2+y^2)]} \iint A\,t(x,y)\, e^{-i\frac{k}{z}(xx'+yy')} dx'd\ldots$$

$$= \frac{1}{i\lambda z} e^{ik[z+\frac{1}{2z}(x^2+y^2)]} A \iint t(x,y)\, e^{-i\frac{k}{z}(xx'+yy')} dx'\ldots$$

$$= \frac{1}{i\lambda z} e^{ik[\quad]} A\, T\left(\frac{kx}{z}, \frac{ky}{z}\right)$$

where $T\left(\frac{kx}{z}, \frac{ky}{z}\right) = \iint t(x,y)\, e^{-\frac{ik}{z}(xx'+yy')} dx$

hence

$$U(x,y,z) = \frac{1}{i\lambda z} e^{ik\left[z+\frac{1}{2z}(x^2+y^2)\right]} A\, T\left(\frac{kx}{z}, \frac{ky}{z}\right)$$

also

$$I(x,y,z) = UU^*$$
$$= \left(\frac{A}{\lambda z}\right)^2 \left| T\left(\frac{kx}{z}, \frac{ky}{z}\right)\right|^2$$

(b)

Fraunhofer diffraction given by Fourier transform

$$FT\{t(x)\} = \int_{-d-L}^{-d+L} e^{-ik_x x} dx + \int_{d-L}^{d+L} e^{-ik_x x} dx$$

where $k_x = \frac{kx}{z}$

$$= \left[\frac{e^{-ik_x x}}{-ik_x}\right]_{-d-L}^{-d+L} + \left[\frac{e^{-ik_x x}}{-ik_x}\right]_{d-L}^{d+L}$$

$$= \left[\frac{e^{-ik_x(-d+L)} - e^{-ik_x(-d-L)}}{-ik_x}\right] + \left[\frac{e^{-ik_x(d+L)} - e^{-ik_x(d-L)}}{-ik_x}\right]$$

$$= +e^{+ik_x d}\left[\frac{e^{-ik_x L} - e^{ik_x L}}{-ik_x}\right] + e^{-ik_x d}\left[\frac{e^{-ik_x L} - e^{ik_x L}}{-ik_x}\right]$$

$$= Le^{ik_x d}\left[\frac{2\sin(k_x L)}{k_x L}\right] + Le^{-ik_x d}\left[\frac{2\sin(k_x L)}{k_x L}\right]$$

$$= 2Le^{ik_x d}\,\mathrm{sinc}(k_x L) + 2Le^{-ik_x d}\,\sin(k_x L)$$

$$= 2L\,\mathrm{sinc}(k_x L)\left[e^{ik_x d} + e^{-ik_x d}\right]$$

$$= 2L\,\mathrm{sinc}(k_x L)\, 2\cos(k_x d)$$

$$= 4L\,\mathrm{sinc}(k_x L)\cos(k_x d)$$

⇒Fraunhofer pattern

$$\therefore \boxed{I(x,y,z) = \left(\frac{4L}{\lambda z}\right)^2 \sin^2(k_x L)\cos^2(k_x d)}, \quad k_x = \frac{kx}{z}$$

(b) also take this:
(and variations)

Fraunhofer diffraction given by Fourier transform

$$T\{t(x)\} = \int_{-d-L}^{-d+L} e^{-ik_x x} dx + \int_{d-L}^{d+L} e^{-ik_x x} dx$$

where $k_x = \frac{kx}{z}$

$$= \left[\frac{e^{-ik_x x}}{-ik_x}\right]_{-d-L}^{-d+L} + \left[\frac{e^{-ik_x x}}{-ik_x}\right]_{d-L}^{d+L}$$

$$= \left[\frac{e^{-ik_x(-d+L)} - e^{-ik_x(-d-L)}}{-ik_x}\right] + \left[\frac{e^{-ik_x(d+L)} - e^{-ik_x(d-L)}}{-ik_x}\right]$$

$$= \frac{1}{k_x}\left[e^{ik_x(d+L)} - e^{ik_x(d-L)} + e^{-ik_x(d-L)} - e^{-ik_x(d+L)}\right]$$

$$= \frac{1}{ik_x}\left\{i2\sin[k_x(d+L)] - i2\sin[k_x(d-L)]\right\}$$

$$= \boxed{\frac{2}{k_x}\left\{\sin[k_x(d+L)] - \sin[k_x(d-L)]\right\}}, \quad k_x = \frac{kx}{z}$$

$$= 2(d+L)\frac{\sin[k_x(d+L)]}{k_x(d+L)} - 2(d-L)\frac{\sin[k_x(d-L)]}{k_x(d-L)}$$

$$= \boxed{2(d+L)\,\mathrm{sinc}[k_x(d+L)] - 2(d-L)\,\mathrm{sinc}[k_x(d-L)]} \quad k_x = \frac{kx}{z}$$

and Intensity $I = \left(\frac{1}{\lambda z}\right)^2 \left|\{FT\}\right|^2$

(c)

approximation made

$$2z \gg k(x'^2 + y'^2)$$

for this case we can make

$$2z \gg kx'^2$$

and here $x' \leq 2(d+L)$

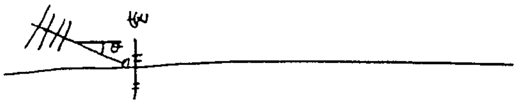
$$\therefore \quad 2z \gg k[2(d+L)]^2$$

$$z \gg \frac{k}{2}\, 4(d+L)^2$$

$$z \gg \frac{2\pi}{\lambda}\, 2(d+L)^2$$

$$z \gg \frac{4\pi}{\lambda}(d+L)^2$$

(d)



$$u(x,y,z) = \frac{1}{i\lambda z} e^{ik\left[z + \frac{1}{2z}(x^2+y^2)\right]} \iint u(x',y',0)\, e^{-i\frac{k}{z}(xx'+yy')} dx'dy'$$

for this case $u(x',y',0) = e^{ik\sin\theta\, x}\, t(x)$

$$u(x,y,z) = \frac{1}{i\lambda z} e^{ik[\quad]} \iint e^{ikx'\sin\theta}\, t(x)\, e^{-i\frac{k}{z}(xx'+yy')} dx'dy'$$

$$= \frac{1}{i\lambda z} e^{ik[\quad]} \int t(x)\, e^{-i\frac{k}{z}(x+z\sin\theta)x'} dx'$$

$$= \frac{1}{i\lambda z} e^{ik[\quad]}\, T\left[\frac{k}{z}(x+z\sin\theta)\right]$$

where $T(\quad)$ is Fourier transform

$\Rightarrow$ diffraction pattern is displaced by $z\sin\theta$

(e)

the fourier plane in image space is at the plane conjugate to the light source, hence, the Fourier plane is at distance d' such that

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f}$$

$$\frac{1}{50\text{cm}} + \frac{1}{d'} = \frac{1}{10\text{cm}}$$

$$\Rightarrow \boxed{d' = 12.5\text{cm}}$$

$\Rightarrow$ Fourier plane is at 12.5 cm to the right of the H' principal plane