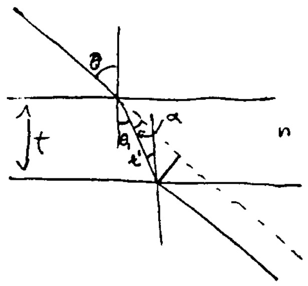
1(a)

PROBLEM 1



$\alpha = \theta - \theta_1$

also $\sin\theta = n\sin\theta_1$

$\theta_1 = \sin^{-1}\left[\frac{\sin\theta}{n}\right]$

now,

$\cos\theta_1 = \frac{t}{t'}$

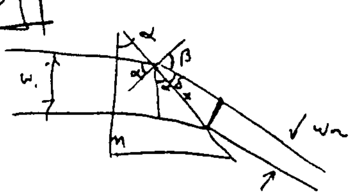
$\Rightarrow \quad t' = \frac{t}{\cos\theta_1}$

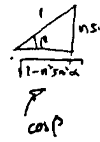
$\boxed{d = t'\sin\alpha}$

$= \frac{t}{\cos\theta_1}\sin\left[\theta - \sin^{-1}\left\{\frac{\sin\theta}{n}\right\}\right]$

$\boxed{d = \frac{t}{\cos\theta_1}\sin\left[\theta - \theta_1\right]}$ where $\boxed{\theta_1 = \sin^{-1}\left[\frac{\sin\theta}{n}\right]}$

1(b)





have, $\cos\alpha = \frac{w_1}{x}$

$\sin\beta = n\sin\alpha \quad \Rightarrow \quad \beta = \sin^{-1}\left[n\sin\alpha\right]$

$\delta = 90 - \beta$

also, $\sin\delta = \frac{w_2}{x}$

$\therefore \quad w_2 = x\sin\delta$

$= \left[\frac{w_1}{\cos\alpha}\right]\sin\delta$

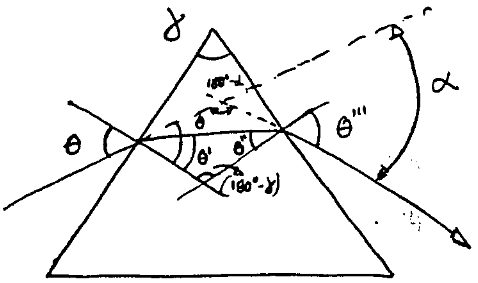
$= \left[\frac{w_1}{\cos\alpha}\right]\sin\left(\frac{\pi}{2} - \beta\right)$

$= \left[\frac{w_1}{\cos\alpha}\right]\sqrt{1 - n^2\sin^2\alpha}$

$\sin(\frac{\pi}{2} - \beta) = \sin(\frac{\pi}{2})\cos\beta - \cancel{\cos(\frac{\pi}{2})\sin\beta}$
$= \cos\beta$

since $\sin^2\beta + \cos^2\beta = 1$

1(c)



from quadrilateral in figure

$180° - \alpha + \theta + \theta''' + 180° - \gamma = 360°$

$\therefore \quad \boxed{\alpha = \theta + \theta''' - \gamma}$

now, from Snell's law,

$n\sin\theta' = \sin\theta$

$\sin\theta' = \frac{1}{n}\sin\theta$

now $\quad \theta' + (\cancel{180} - \gamma) + \theta'' = \cancel{180}$

$\theta'' = \gamma - \theta'$

$n\sin\theta'' = \sin\theta'''$

$n\sin[\gamma - \theta'] = \sin\theta'''$

$\sin\theta''' = n\sin[\gamma - \theta']$

$= n\sin(\gamma)\cos(\theta') - n\cos(\gamma')\sin(\theta')$

$= n\sin(\gamma)\cos(\theta') - n\cos(\gamma)\sin(\theta')$

$\sin\theta' = \frac{1}{n}\sin\theta$

$\cos\theta' = \sqrt{1 - \frac{1}{n^2}\sin^2\theta}$

$\boxed{\sin\theta''' = \left[n\sin(\gamma)\sqrt{1 - \frac{1}{n^2}\sin^2\theta}\right] - n\cos(\gamma)\frac{1}{n}\sin(\theta)}$

$\therefore \quad \cos\theta''' = \sqrt{1 - \sin^2\theta'''}$

$\therefore \quad \theta''' = \sin^{-1}\left\{\left[n\sin(\gamma)\sqrt{1 - \frac{1}{n^2}\sin^2\theta}\right] - \cos(\gamma)\sin(\theta)\right\}$

$\therefore$

$$\alpha = \theta + \theta''' - \gamma$$

where θ''' given in last page.

to find minimum α then must take derivative and equate to zero.

ie must solve $\dfrac{\partial \alpha}{\partial \theta} = 0$

If they get this far then they get full credit!
(it gets very messy from here!)

Problem 2

$\boxed{2a}$ using $\frac{1}{o} + \frac{1}{i} = \frac{1}{f}$

for first lens have

$$\frac{1}{20} + \frac{1}{i} = +\frac{1}{10}$$

$$\therefore \boxed{i = 20}$$

magnification is $\beta_1 = -\frac{20}{20} = -1$

for second lens have

$o = -5$ (since $i = 20$ above and separation between lenses is 15 cm)

$$\therefore \frac{1}{-5} + \frac{1}{i} = \frac{1}{-10}$$

$$\therefore \frac{1}{i} = -\frac{1}{10} + \frac{1}{5}$$

$$\therefore \boxed{i = 10}$$

magnification is $\beta_2 = \frac{-10}{-5} = +2$

MATRIX APPROACH

$$\begin{pmatrix} 1 & 0 \\ -1/f_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ -1/15 & 1 \end{pmatrix}\begin{pmatrix} 1 & 20 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/10 & 1 \end{pmatrix}\begin{pmatrix} 1 & 15 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/10 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 20 \\ -1/15 & -\frac{20}{15}+1 \end{pmatrix}\begin{pmatrix} 1 & 15 \\ \frac{1}{10} & \frac{15}{10}+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{10} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 20 \\ -\frac{1}{15} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 15 \\ \frac{1}{10} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{10} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 20 \\ -\frac{1}{15} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 - \frac{15}{10} & 15 \\ \frac{1}{10} - \frac{1}{4} & \frac{5}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 20 \\ -\frac{1}{15} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 15 \\ -\frac{3}{20} & \frac{5}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} - 3 & 15 + 50 \\ +\frac{1}{30} + \frac{1}{20} & -1 - \frac{5}{6} \end{pmatrix}$$

$$M_{vv'} = \begin{pmatrix} -\frac{7}{2} & 65 \\ \frac{1}{12} & -\frac{11}{6} \end{pmatrix}$$

if object is 20 cm away then

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -7/2 & 65 \\ \frac{1}{12} & -\frac{11}{6} \end{pmatrix}\begin{pmatrix} 1 & 20 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & -70 + 65 \\ \frac{1}{12} & \frac{20}{12} - \frac{11}{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & -5 \\ \frac{1}{12} & -\frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{7}{2} + \frac{1}{12}d & -5 - \frac{1}{6}d \\ \frac{1}{12} & -\frac{1}{6} \end{pmatrix}$$

hence, system matrix is

$$M = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} + \frac{1}{12}d & -5 - \frac{1}{6}d \\ \frac{1}{12} & -\frac{1}{6} \end{pmatrix}$$

for imaging have $B' = 0$ ...

$$\therefore \quad -5 - \frac{1}{6}d = 0$$

$$-5 = \frac{1}{6}d$$

$$\therefore \boxed{d = -30\,\text{cm}}$$

MAGNIFICATION IS GIVEN BY $+A'$

$$\therefore \quad -\frac{7}{2} + \frac{1}{12}d = M$$

$$\therefore \boxed{M = -6}$$

for third lens have $o = 10$
(from last result and since second and third lenses separated by 20 cm)

$$\therefore \quad \frac{1}{o} + \frac{1}{i} = \frac{1}{15}$$

$$\frac{1}{i} = \frac{1}{15} - \frac{1}{10}$$

$$\therefore \boxed{i = -30}$$

magnification is $\beta_3 = -\frac{-30}{10} = +3$

and $\therefore$ this is distance from last lens to find image

$$\Rightarrow \boxed{i = -30\,\text{cm}}$$

total magnification is

$$\beta_{TOT} = \beta_1 \beta_2 \beta_3$$

$$= (-1)(+2)(+3)$$

MAGNIFICATION $= -6$

2(b)

vertical polarization $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

polarizer at 45° $\quad P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$\lambda/4$ plate $\quad M = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$\therefore$ output polarization is

$$= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$\boxed{= \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$ ie circular polarization

Problem 3

3(a)

wave in one arm of interferometer

$$U_1 = e^{ikr}$$

wave propagating in other arm of interferometer

$$U_2 = e^{ik(r+2d)}$$

$\Rightarrow$ total E-field is

$$E_{TOT} = U_1 + U_2$$

$$\boxed{E_{TOT} = e^{ikr} + e^{ik(r+2d)}}$$

$\boxed{3(b)}$

Intensity given by $E_{TOT} E_{TOT}^*$

$$I = [e^{ikr} + e^{ik(r+2d)}][e^{-ikr} + e^{-ik(r+2d)}]$$

$$= 1 + 1 + e^{i[kr - kr - 2kd]} + e^{-i[kr - kr - k2d]}$$

$$= 2 + 2\cos(2kd)$$

$$\boxed{I \propto 1 + \cos(2kd)}$$

it is ok to ignore the 2 here

note, the optical path length is

$$\Delta OPL = 2kd$$
$$= 2\,\frac{2\pi}{\lambda}\,d$$

if originally have constructive interference, then will get destructive interference when the optical path length changes by $\frac{\pi}{2}$.

The OPL changed by changing λ, is then

$$2\,\frac{2\pi}{\lambda}d - 2\,\frac{2\pi}{(\lambda+\Delta\lambda)}\,d = \frac{\pi}{2}$$

$$\frac{\lambda+\Delta\lambda-\lambda}{\lambda(\lambda+\Delta\lambda)} = \frac{1}{8d}$$

$$\Delta\lambda = \frac{1}{8d}[\lambda^2 + \lambda\Delta\lambda]$$

$$\Delta\lambda - \frac{1}{8d}\lambda\Delta\lambda = \frac{\lambda^2}{8d}$$

$$\Delta\lambda\left[1 - \frac{\lambda}{8d}\right] = \frac{\lambda^2}{8d}$$

$$\boxed{\Delta\lambda = \frac{\lambda^2}{8d\left[1 - \frac{\lambda}{8d}\right]}}$$

and if d is very large, then

$$\Delta\lambda \approx \frac{\lambda^2}{8d}$$

Problem 4

$\boxed{4\text{-}a}$ $$A(k_x, k_y; 0) = \iint U(x,y,0)\, e^{-i(k_x x + k_y y)}\, dx\,dy$$

$\boxed{4\text{-}b}$ $$A(k_x, k_y; z) = \iint U(x,y,z)\, e^{-i(k_x x + k_y y)}\, dx\,dy$$

$\boxed{4\text{-}c}$ from (4-b) have

$$U(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \iint A(k_x, k_y; z)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

however $U(x,y,z)$ must satisfy Helmholtz eq

$$\nabla^2 U + k^2 U = 0$$

hence, by plugging into eq. get

(recall $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U + k^2 U = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[\left(\frac{1}{2\pi}\right)^2 \iint A(k_x, k_y; z)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y\right]$$
$$+ k^2 \left[\left(\frac{1}{2\pi}\right)^2 \iint A(k_x, k_y; z)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y\right] =$$

$$\frac{d^2 A}{dz^2} - k_x^2 A(k_x, k_y; z) - k_y^2 A(k_x, k_y; z) + k^2 A(k_x, k_y; z) = 0$$

$$\therefore\ \frac{d^2}{dz^2} A(k_x, k_y; z) + [k^2 - k_x^2 - k_y^2] A(k_x, k_y; z) = 0$$

$\Rightarrow$ set

$$\frac{d^2}{dz^2} A(k_x, k_y; z) + \left[k^2 - k_x^2 - k_y^2\right] A(k_x, k_y; z) = 0$$

and solution to this eq is

$$\boxed{A(k_x, k_y; z) = A(k_x, k_y; 0)\, e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, z}}$$

(4-d)

Convergent wave $U_d(x,y,z) = A_0 \frac{d}{d-z}\, e^{ik\left[z - \frac{1}{2(d-z)}(x^2+y^2)\right]}$

and at $z=0$ (object plane)

$$U_0(x,y,0) = U_0(x,y,0)\, t(x,y)$$
$$= A_0\, e^{i\frac{k}{2d}(x^2+y^2)}\, t(x,y)$$

hence, from Fresnel's integral

$$U(x,y,z) = \frac{1}{i\lambda z}\, e^{ik\left[z + \frac{1}{2z}(x^2+y^2)\right]} \iint A_0\, e^{-\frac{ik}{2d}(x'^2+y'^2)}\, t(x',y') \cdot e^{-i\frac{k}{z}(xx'+yy')}\, e^{i\frac{k}{2z}(x'^2+y'^2)}\, dx'dy'$$

$$= \frac{1}{i\lambda z}\, e^{ik[\quad]} \iint t(x',y')\, e^{-i\frac{k}{z}(xx'+yy')}\, e^{i\frac{k}{2}\left[\frac{1}{z} - \frac{1}{d}\right](x'^2+y'^2)}\, dx$$

and for $z=d$ this reduces to

$$U(x,y,d) = \frac{A_0}{i\lambda d}\, e^{ik[\quad]} \iint t(x',y')\, e^{-i\frac{k}{d}(xx'+yy')}\, dx'dy'$$

$$\boxed{= \frac{A_0}{i\lambda d}\, e^{ik\left[d + \frac{1}{2d}(x^2+y^2)\right]}\; FT\{t(x,y)\}}$$

$/k_x$.