SIF4040 Optics, Continuation Exam, July 31, 2001

Suggested Solution

Problem 1

a) $\mathcal{P}_1 = \frac{n'-n}{R} = \frac{1}{3R}$; $\mathcal{P}_2 = \frac{n-n'}{(-R)} = \mathcal{P}_1 = \frac{1}{3R}$

$t = 2R$

$$\mathbb{M}_{VV'} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3R} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2}R \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{3R} & 1 \end{pmatrix}$$

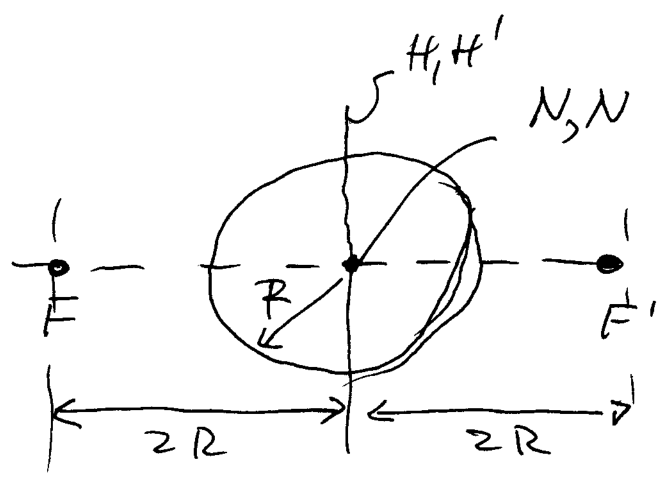
$$= \begin{pmatrix} \frac{1}{2} & \frac{3}{2}R \\ -\frac{1}{2R} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Since the index is $n=1$ on both sides, we have $f = -\frac{1}{C} = \underline{2R}$ QED.

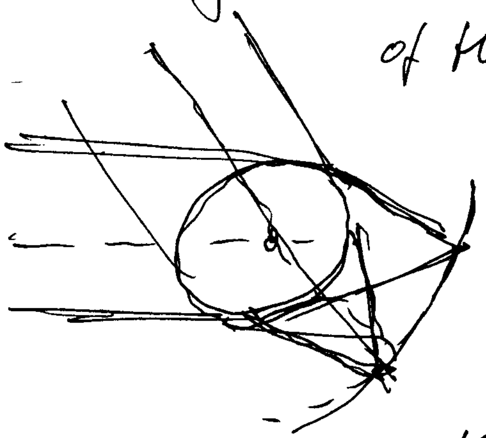
b) $h = \frac{n}{C}(1-D) = \underline{-R}$ $\quad (n = n' = 1)$

$h' = \frac{n'}{C}(1-A) = \underline{-R}$

The principal planes are virtual planes through the center of the source. Since $n = n' = 1$, the nodal points are in the principal planes. (Only rays through the center emerges on the other side without a directional change)



c) All axes through the center are equivalent. If the solar image is to be focused equally well for all positions of the sun, the screen must be concentric with the sphere and have the radius $f = 2R = \underline{30\ cm}$



The image of the sun has the diameter

$$0.5^\circ \cdot \frac{\pi}{180} \cdot 2R = \frac{\pi}{360}\, 0.3\ m = \underline{2.62\ mm}$$

d) Focused power = incident power on area $\pi\left(\frac{R}{2}\right)^2$

$$1\ \frac{kW}{m^2} \cdot \pi\left(\frac{R}{2}\right)^2 = \pi\,(0.075)^2\ kW = \underline{17.67\ W}$$

Irradiance in the image

$$\frac{\text{Focused power}}{\text{Image area}} = \frac{17.67\ W}{\pi\left(\frac{0.00262}{2}\right)^2 m^2} = \underline{3277.5\ \frac{kW}{m^2}}$$

Problem 2

a) Optical path-length difference

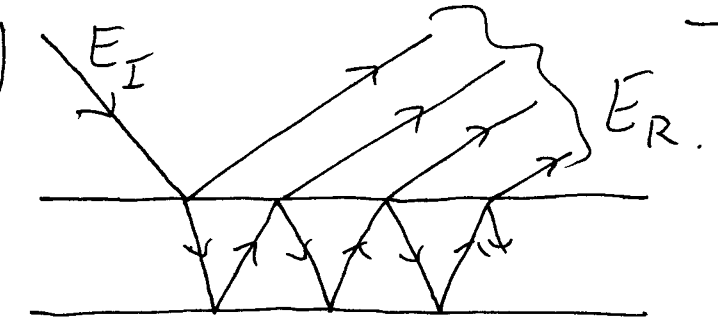
$\Delta = n_f (AB + BC) - n_0 AD$, where

$$\left.\begin{aligned} &AB = BC = t/\cos\theta_t \\ &AD = AC\sin\theta_i = 2AG\sin\theta_i = \\ &= 2t\tan\theta_t \sin\theta_i \end{aligned}\right\} \Rightarrow$$

$$\Delta = 2t\left(\frac{n_f}{\cos\theta_t} - n_0 \sin\theta_i \, tg\,\theta_t\right); \quad \text{Snell: } n_0\sin\theta_i = n_f \sin\theta_t$$

$$= 2tn_f\left(\frac{1}{\cos\theta_t} - \frac{\sin^2\theta_t}{\cos\theta_t}\right) = \underline{2tn_f\cos\theta_t} \quad \text{QED.}$$

Phase difference $\underline{\delta = k\Delta = 4\pi \frac{t}{\lambda} n_f \cos\theta_t}$

b) 

The reflected wave is the sum of the surface reflection and infinitely many multiply reflected internal reflections. The phase difference between each of these are δ (computed in a)). By summing an infinite geometric series and making use of expressions for the transmission and reflection coefficients at each surface, one obtains an expression for $\frac{E_R}{E_i}$ and the given $\frac{I_R}{I_I} = \left| \frac{E_R}{E_I} \right|^2$.

c) Energy conservation: $\frac{I_R}{I_I} + \frac{I_T}{I_I} = 1 \Rightarrow$

$$T = \frac{I_T}{I_I} = 1 - \frac{I_R}{I_I} = 1 - \frac{2r^2(1-\cos\delta)}{1+r^4-2r^2\cos\delta}$$

$$= \frac{1+r^4-2r^2}{1+r^4-2r^2\cos\delta} = \frac{1}{1 + \frac{2r^2}{1+r^4-2r^2}\underbrace{(1-\cos\delta)}_{=2\sin^2\frac{\delta}{2}}}$$

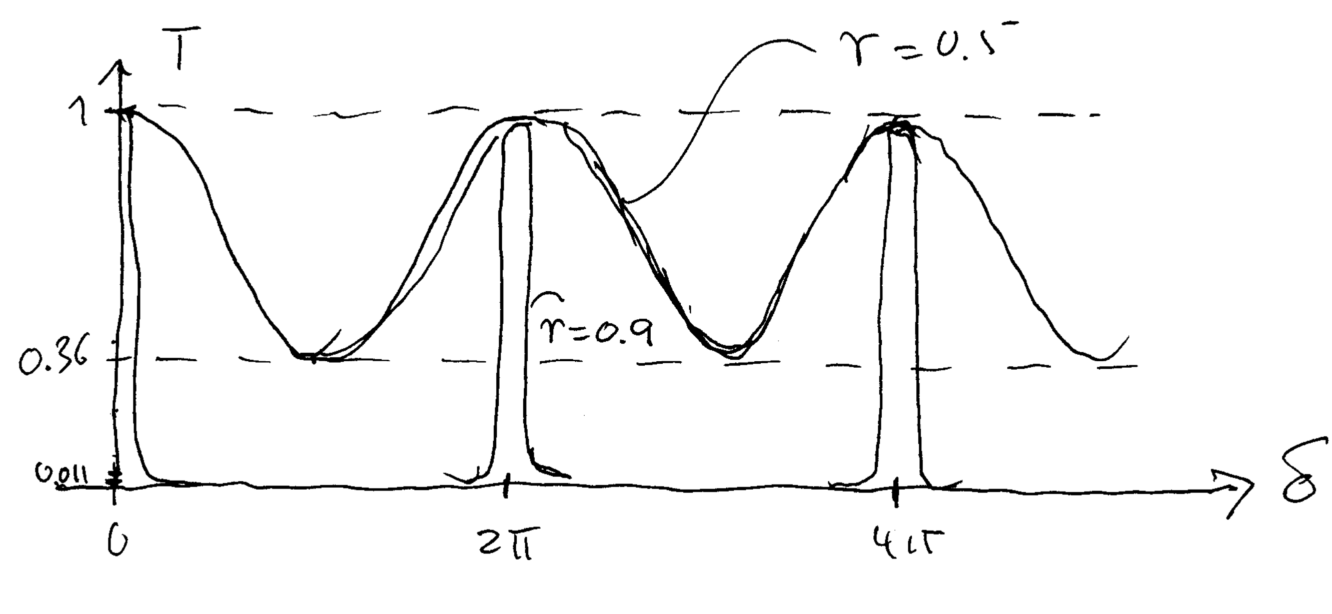
$$= \frac{1}{1+\frac{4r^2}{1+r^4-2r^2}\sin^2\frac{\delta}{2}} = \underline{\frac{1}{1+F\sin^2\frac{\delta}{2}}} \qquad \text{QED}$$

where

$$F = \frac{4r^2}{1+r^4-2r^2} = \underline{\underline{\left(\frac{2r}{1-r^2}\right)^2}}$$

$r = 0.5 \Rightarrow F = \left(\frac{1}{0.75}\right)^2 = 1.777$, $T_{min} = \frac{1}{1+F} = \underline{0.36}$

$r = 0.9 \Rightarrow F = \left(\frac{1.8}{0.19}\right)^2 = 89.75$, $T_{min} = 0.011$



d) $\Delta\delta = 2\pi$; $T = \frac{1}{2}$ for $1 + F\sin^2\left(\frac{\delta_{1/2}}{4}\right) = 2$,

i.e., for $\sin^2\frac{\delta_{1/2}}{4} = \frac{1}{F} \ll 1$ for $F \gg 1$

$$\Rightarrow \delta_{1/2}^2 = 16/F \Rightarrow \delta_{1/2} = 4/\sqrt{F}$$

Finesse: $\frac{\Delta\delta}{\delta_{1/2}} = \frac{\pi}{2}\sqrt{F} = \pi\frac{r}{1-r^2} = \underline{\underline{156.3}}$

for $r = 0.99$.

Problem 3

a) Plane wave $A_0 e^{i\vec{k}\cdot\vec{r}} = A_0 e^{i(ux+vy+wz)}$, $w = \sqrt{k^2-u^2-v^2}$

At $z=0$: $A_0 e^{i(ux+vy)}$ (1)

Here:

$E(x,y,0) = t(x,y)A = \frac{1}{2}A\left(1+\cos\left(2\pi\frac{x}{a}\right)\right)$

$= \frac{1}{2}A + \frac{1}{4}Ae^{i2\pi\frac{x}{a}} + \frac{1}{4}Ae^{-i2\pi\frac{x}{a}}$

= 3 terms of the same type as in (1). Give rise to 3 plane waves behind the grating:

$E(x,y,z) = E^{(0)}(x,y,z) + E^{(1)}(x,y,z) + E^{(-1)}(x,y,z)$,

where

$$E^{(0)}(x,y,z) = \frac{A}{2} e^{ikz}$$ Plane wave in z-direction

$$E^{(1)}(x,y,z) = \frac{A}{4} e^{i\left(2\pi\frac{x}{a} + z\sqrt{k^2-(2\pi/a)^2}\right)}$$

$$E^{(-1)}(x,y,z) = \frac{A}{4} e^{i\left(-2\pi\frac{x}{a} + z\sqrt{k^2-(2\pi/a)^2}\right)}$$

Here $k = 2\pi/\lambda \gg \frac{2\pi}{a}$ since $\lambda = 500\,nm \ll a = 0.01\,mm$

I.e., $E^{(\pm 1)}$ are propagating at ~~the~~ angles

$\pm \frac{2\pi}{ka} = \pm\frac{\lambda}{a} = \sin\theta \approx \theta$ with the axis.

b) Each of the three plane waves are focused at three points in the back focal plane of the lens.

$E^{(0)}$ is focused on-axis, at $x=0, y=0$

$E^{(\pm 1)}$ are focused at $x = f\theta = \pm\lambda\frac{f}{a} = \underline{\pm 1\,cm}$

and $\underline{y=0}$.

c) Since the object distance is $s = 2f = 40\,cm$, the lens formula yields $\underline{s' = 2f = 40\,cm}$ and the magnification $\underline{\beta = -\frac{s'}{s} = -1}$.

In the image plane the intensity distribution is

$$I(x,y) = \frac{A^2}{4}\left(1+\cos\left(2\pi\frac{(-x)}{a}\right)\right)^2 = \underline{\frac{A^2}{4}\left(1+\cos\left(2\pi\frac{x}{a}\right)\right)^2}$$

because $\cos(-\alpha) = \cos\alpha$.

d) Nothing happens to the intensity distribution in the back focal plane because a displacement of the object only yields a phase-change in the Fraunhofer diffraction pattern (the Fourier shift-theorem). The displacement does not influence the directions of the three plane waves.

For the intensity distribution in c) we get a displacement $-\Delta x$ because the magnification is $\beta = -1$.