Exam, May 24, 2002: Suggested solution

Problem 1

a) System matrix:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \underline{\begin{pmatrix} 1-\frac{t}{f_1} & t \\ -\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{t}{f_1 f_2}\right) & 1-\frac{t}{f_2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}}.$$

The focal length of the composite system is $f = -1/C$, which yields

$$\frac{1}{f} = \frac{1}{f_1}+\frac{1}{f_2}-\frac{t}{f_1 f_2} \quad \text{and} \quad \underline{t = f_1 + f_2 - \frac{f_1 f_2}{f} = 12.5\,\text{cm}.}$$

b) The principal planes H and H' are conjugate planes (images of each other) for which the magnification is: $\underline{\beta = +1}$.

From the given formulas we obtain:

$$h = n(1-D)/C = -\frac{t}{f_2}f = \underline{-\frac{tf_1}{f_1+f_2-t} = 4.1666\,\text{m}}. \quad \text{(Distance from } H \text{ to } L_1\text{)}$$

$$h' = n'(1-A)/C = -\frac{t}{f_1}f = \underline{-\frac{tf_2}{f_1+f_2-t} = -83.333\,\text{cm}}. \quad \text{(Distance from } L_2 \text{ to } H'\text{)}$$

c) The image plane is in <u>the back focal plane</u> of the composite system. It is located at a distance

$$h'+f = \underline{\left(1-\frac{t}{f_1}\right)f = 16.666\,\text{cm}} \quad \text{to the right of } L_2.$$

The physical length of the system (from L_1 to the image plane) is

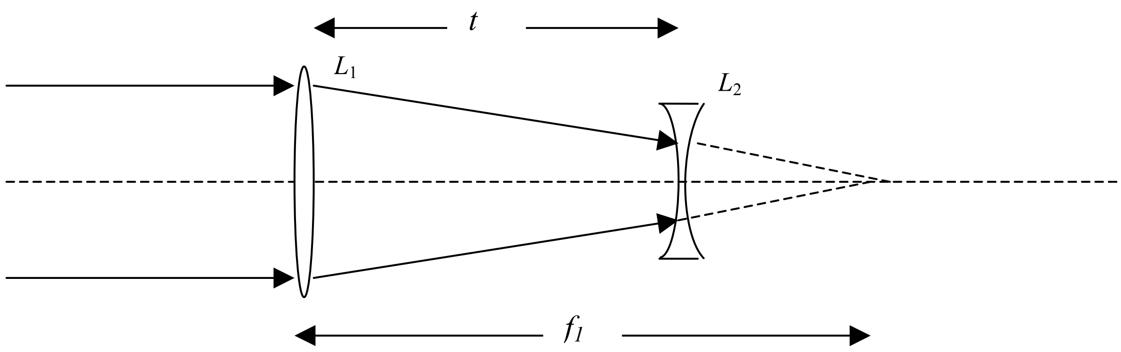
$$h'+f+t = \underline{\left(1-\frac{t}{f_1}\right)f + t = 29.166\,\text{cm}}.$$

The physical length of this system is less than a third of the focal length, so this is a <u>telefocus lens.</u>

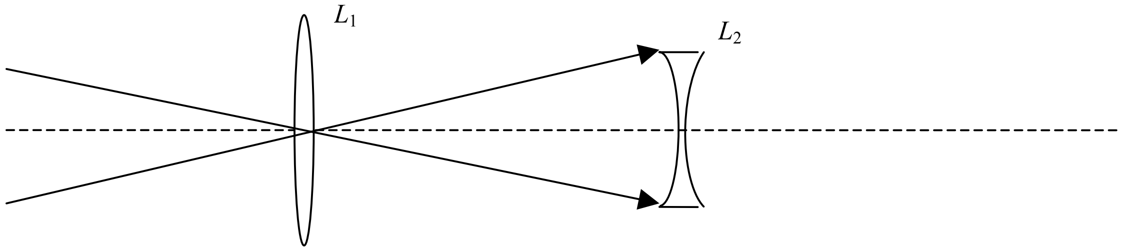
d)

- The <u>aperture stop</u> is the physical stop that limits the ray bundle from an object point on axis to the corresponding image point.
- The <u>entrance pupil</u> is the aperture stop imaged to the object space (i.e., seen from the object side).
- The <u>exit pupil</u> is the aperture stop imaged to the image space (i.e., seen from the image side).
- A <u>chief ray</u> is the ray from an object point to the corresponding image point that passes through the center of the aperture stop (and the associated pupils).
- The <u>field stop</u> is the stop that limits the bundle of chief rays from the object to the image.

- The <u>entrance window</u> is the field stop imaged to the object space (i.e., seen from the object side).
- The <u>exit window</u> is the field stop imaged to the image space (i.e., seen from the image side).



For the object point on axis we now have the situation illustrated in the figure above and, from the given dimensions and the results above, we immediately see that <u>L_1 is the aperture stop</u>. Since we only have two stops, <u>L_2 is the field stop.</u>



For the bundle of chief-rays that limits the angular field of view, we have the situation illustrated above. The angular diameter of the field of view is therefore equal to the angle subtended by L_2 at L_1: $\frac{D_2}{t} = \frac{1}{12.5}\frac{180^0}{\pi} = 4.594^0$.

Problem 2

a) For a monochromatic point source: $I(s) = 2I_0[1+\cos(ks)] = 2I_0\left[1+\cos\left(\frac{\omega s}{c}\right)\right]$,

where $c = \omega/k$ is the velocity of light. For a polychromatic point source, the contribution the from the spectral range $d\omega$ is

$dI(s) = 2dI_0\left[1+\cos\left(\frac{\omega s}{c}\right)\right] = 2W(\omega)\left[1+\cos\left(\frac{\omega s}{c}\right)\right]d\omega$, which yields:

$$I(s) = 2\int_0^\infty W(\omega)\left[1+\cos\left(\frac{\omega s}{c}\right)\right]d\omega.$$

From the given formula for $\Gamma(\tau)$, we have

$\Gamma(0) = \int_0^\infty W(\omega)d\omega$ and $\mathrm{Re}\,\Gamma(s/c) = \int_0^\infty W(\omega)\cos(\omega s/c)d\omega$, which directly yields the given formula. QED!

b) Direct integration yields

$$\Gamma(\tau) = \frac{I_0}{\Delta\omega}\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2}\exp(-i\omega\tau)d\omega = \frac{I_0}{\Delta\omega}\exp(-i\omega_0\tau)\frac{\exp(-i\Delta\omega\tau/2)-\exp(i\Delta\omega\tau/2)}{-i\tau}$$

$$= \frac{I_0}{\Delta\omega}\exp(-i\omega_0\tau)\frac{2\sin(\Delta\omega\tau/2)}{\tau} = \underline{I_0\exp(-i\omega_0\tau)\mathrm{sinc}(\Delta\omega\tau/2)}.$$

Direct substitution into the formula in a) yields the given expression. QED!

c) The visibility is defined as $V \equiv \frac{I_{max}-I_{\min}}{I_{max}+I_{\min}}$, where I_{max} and I_{min} are, respectively, the maximum and minimum intensity as we move from one fringe to the next by changing the path-length difference.

In the interference signal $I(s) = 2I_0\left[1+\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\cos(\omega_0\, s/c)\right]$, the cosine term describes the interference fringes while the sinc term is a slowly varying envelope that determines the visibility. We then have

$I_{max} = 2I_0\left[1+\left|\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\right|\right]$ and $I_{mim} = 2I_0\left[1-\left|\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\right|\right]$, from which we obtain the visibility function:

$$V(s) = \frac{I_{max}-I_{\min}}{I_{max}+I_{\min}} = \underline{\left|\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\right|}.$$

With $\omega_0 = 2\pi f_0 = 2\pi c/\lambda_0 = 10\Delta\omega$ we have $\Delta\omega = 2\pi c/(10\lambda_0)$ and directly obtain the given formula. QED!

d) The first zero of the visibility function is at $\frac{\Delta\omega\, s}{2c} = \pi$, which yields for the longitudinal coherence length: $l_c = \underline{2\pi c/\Delta\omega}$.

When we go from one fringe to the next, *s* is changed by the amount: $2\pi c/\omega_0 = \lambda_0$.

The number of interference fringes that can be observed in the range $-l_c/2 \le s \le l_c/2$ is therefore: $l_c/\lambda_0 = \underline{\omega_0/\Delta\omega}$.

Problem 3

a) The field in the plane $z = 0$, directly behind the grating, is

$$U(x,y,0) = t(x,y)U_0(0) = \frac{A}{2}\left[1+\cos(2\pi x/a)\right]$$
$$= \frac{A}{2} + \frac{A}{4}\exp(i2\pi x/a) + \frac{A}{4}\exp(-i2\pi x/a). \qquad (1)$$

For a plane wave, we have: $A\exp(i\mathbf{k}\cdot\mathbf{r}) = A\exp\left[i\left(ux+vy+z\sqrt{k^2-u^2-v^2}\right)\right]$

In the plane $z = 0$ it reduces to: $A\exp\left[i(ux+vy)\right]$.

We see that the field behind the grating is the sum of three terms of precisely this type with $v = 0$ and $u = 0, 2\pi/a, -2\pi/a$, respectively.

The field behind the grating therefore consists of three plane waves propagating in different directions, one for each term in (1):

$U(x,y,z) = U_1(x,y,z) + U_2(x,y,z) + U_3(x,y,z)$, where

$$U_1 = \frac{A}{2}\exp(ikz),$$
$$U_2 = \frac{A}{4}\exp(i2\pi x/a + z\sqrt{k^2-(2\pi/a)^2}),$$
$$U_3 = \frac{A}{4}\exp(-i2\pi x/a + z\sqrt{k^2-(2\pi/a)^2}).$$

Direct addition yields the given formula. QED!

b) The field can be written as

$$U(x,y,z) = \frac{A}{2}\left[\exp(ikz) + \cos(2\pi x/a)\exp\left(iz\sqrt{k^2-(2\pi/a)^2}\right)\right]$$
$$= U_0(z)\frac{1}{2}\left\{1+\cos(2\pi x/a)\exp\left[i\left(\sqrt{k^2-(2\pi/a)^2}-k\right)z\right]\right\}$$

Perfect images of the grating are found at distances for which $z(k-\sqrt{k^2-(2\pi/a)^2}) = 2\pi m$, where m is an integer, i.e., for

$$z = z_m = \frac{2\pi m}{k-\sqrt{k^2-(2\pi/a)^2}} = \frac{m\lambda}{1-\sqrt{1-(\lambda/a)^2}}, \quad m = 1, 2, 3,\ldots \text{etc.}$$

In the paraxial approximation we have $2\pi/a \ll k$, i.e., $\lambda \ll a$, and obtain the given formula from $1-\sqrt{1-(\lambda/a)^2} \approx \frac{1}{2}(\lambda/a)^2$. QED!

c) The three plane waves propagate at different angles with the axis: U_1 is propagating in the z direction, but U_2 and U_3 are propagating at the angles $\pm\theta$, where $k\sin\theta = 2\pi/a$, i.e., $\sin\theta = \lambda/a$. With the given numerical values we have $\sin\theta \approx \theta = \lambda/a$. They are focused at three different points in the back focal plane of the lens:

- U_1 is focused on axis with the intensity $A^2/4$.
- U_2 and U_3 are focused at $y = 0$ and $x = \pm f\theta = \pm f\lambda/a = \underline{\pm 1\,\text{cm}}$ with intensities $A^2/16$.

d) If the grating is displaced sideways, the intensity distribution in the back focal plane remains unchanged.

A sideways displacement of the grating will give opposite phase shifts of U_2 and U_3 but does not influence their directions of propagation. The three waves are therefore focused at the same points in the back focal plane and with unchanged intensities. This is an example of the Fourier shift-theorem.