Suggested solution - Exam, May 27, 2003

Problem 1

a) System matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{t}{f_1} & t \\ -\left(\dfrac{1}{f_1}+\dfrac{1}{f_2}-\dfrac{t}{f_1 f_2}\right) & 1-\dfrac{t}{f_2} \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 1/3 & 16\,\text{cm} \\ -1/48\,\text{cm}^{-1} & 2 \end{pmatrix}}$$

Power: $P = -C = 1/48\ \text{cm}^{-1} = \underline{2.08\ \text{m}^{-1}\text{(diopter)}}$

b) From a) and $n = n' = 1$: $f = f' = -n/C = \underline{48\ \text{cm}}$. $h = (1-D)n/C = \underline{48\ \text{cm}}$, i.e., H is 48 cm to the left of L_1. $h' = (1-A)n'/C = \underline{-32\ \text{cm}}$, i.e., H' is 32 cm to the left of L_2. The front focal point F is a distance $f+h$ = 96 cm to the left of L_1. The back focal point F' is a distance $f+h'$ = 16 cm to the right of L_2

c) The object is a distance s = 144 cm – 48 cm = $2f$ to the left of H. From the lens formula we then see that the image is the same distance $s' = 2f$ to the right of H'. The image is located the distance $h' + 2f = \underline{64\ \text{cm}}$ to the right of L_2. The magnification is $\beta = -s'/s = \underline{-1}$.

d) *The aperture stop* is the physical stop that limits the ray-bundle contributing to the image point on-axis.

The field stop is the physical stop that limits the bundle of chief-rays through the center of the aperture stop.

The entrance pupil is the aperture stop seen from or imaged into object space.

The exit pupil is the aperture stop seen from or imaged into image space.

The entrance window is the field stop seen from or imaged into object space.

T*he exit window* is the field stop seen from or imaged into image space.

We image all stops to the object space (might as well have chosen the image space): L_1 is in object space (it is imaged onto itself). At the object point on-axis its aperture subtends the angle 2/144 rad = 1/72 rad. The image of L_2 (formed by imaging through L_1) is a distance s to the left of L_1, where the lens formula: $1/s + 1/(16\ \text{cm}) = 1/f_1$ yields s = –48 cm. The image is therefore located 48 cm to the right of L_1 and the magnification is 48/16 = 4. The image of the aperture of L_2 therefore has a diameter of 16 cm and it subtends the angle 8/(144+48) rad = 1/21 rad at the object point on axis. *The smallest angle is subtended by L_1, which is therefore the* *aperture stop.*

(This can also be seen directly: the object distance is larger than f_1 and then L_1 will serve to *converge* the ray-bundle from the object point on-axis so that it can never be limited by L_2).

The exit pupil is the aperture stop L_1 imaged into image space. Then L_1 is imaged by L_2 to a virtual image at a distance 8 cm to the left of L_2 and with the magnification 8/16=1/2. *The exit pupil* is therefore 8 cm to the left of L_2 and has a diameter of 2 cm.

F-number in image space: $F' = (64+8)/2 = \underline{36.}$

Problem 2

a) With a 50/50 beam-splitter each of the two waves has the same intensity $I_1=I_2=I_0$, and from the interference equation we then have

$$I(s) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(ks) = 2I_0[1-\cos(ks)] = 2I_0[1+\cos(\omega s/c)].$$

where we have used that $k=\omega/c$.

For a polychromatic source the corresponding contribution to the intensity from the frequency range $d\omega$ is: $dI(s) = 2W(\omega)[1+\cos(\omega s/c)]d\omega$. Integrating over all frequencies we obtain:

$$I(s) = 2\int_0^\infty W(\omega)[1+\cos(\omega s/c)]d\omega .$$

From the given formulas we have

$\Gamma(0) = \int_0^\infty W(\omega)d\omega$ and $\mathrm{Re}\,\Gamma(s/c) = \int_0^\infty W(\omega)\cos(\omega s/c)d\omega$, which directly yields

$$I(s) = 2[\Gamma(0) + \mathrm{Re}\,\Gamma(s/c)].\ \text{QED}$$

b) The visibility is defined as $V = \frac{I_{max} - I_{\min}}{I_{max} + I_{\min}}$, where $I_{\max}$ is the maximum intensity (at constructive interference) and $I_{\min}$ is the minimum intensity (at destructive interference) of the interference signal.

With the given interference signal, we have $I_{max} = 2I_0[1+\exp(-|s|/L)]$ and $I_{\min} = 2I_0[1-\exp(-|s|/L)]$, which yield: $V(s) = \exp(-|s|/L)$.

By direct substitution, we see that the given interference signal is obtained from

$$\Gamma(s/c) = I_0 \exp\left[-\left(|s|/L + i2\pi s/\lambda_0\right)\right], \text{ i.e., } \Gamma(\tau) = I_0 \exp\left[-\left(|\tau|c/L + i2\pi\tau c/\lambda_0\right)\right],$$

which equals the given expression for

$$T = L/c = 1.66666 \cdot 10^{-11} \text{ sec} = 16.6666 \text{ ps}$$

and

$$\omega_0 = 2\pi c/\lambda_0 = 3.77 \cdot 10^{15} \text{ rad/s.}$$

c) We have $\Gamma(\tau) = \Gamma(0)\gamma(\tau) = I\gamma(\tau)$, where $I = \Gamma(0)$ is the total intensity (a constant). The given formulas then yield

$$W(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\tau) \exp(i\omega\tau) d\tau$$

$$= \frac{I_0}{2\pi} \int_{-\infty}^{\infty} \gamma(\tau) \exp(i\omega\tau) d\tau = \frac{I_0}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\left(|\tau|/T + i(\omega_0 - \omega)\tau\right)\right] d\tau \; .$$

Since we have $|\tau| = \begin{cases} \tau & \text{for } \tau > 0 \\ -\tau & \text{for } \tau < 0 \end{cases}$, the integral must be subdivided into two parts:

$$W(\omega) = \frac{I_0}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\left(|\tau|/T + i(\omega_0 - \omega)\tau\right)\right] d\tau$$

$$= \frac{I_0}{2\pi} \int_{-\infty}^{0} \exp\left[\left(1/T - i(\omega_0 - \omega)\right)\tau\right] d\tau + \frac{I_0}{2\pi} \int_{0}^{\infty} \exp\left[-\left(1/T + i(\omega_0 - \omega)\right)\tau\right] d\tau$$

$$= \frac{I_0}{2\pi} \int_{0}^{\infty} \left\{\exp\left[-\left(1/T - i(\omega - \omega_0)\right)\tau\right] + \exp\left[-\left(1/T + i(\omega - \omega_0)\right)\tau\right]\right\} d\tau,$$

where the first term in the last line follows by a simple change of the integration variable ($\tau \to -\tau$). Direct integration yields

$$W(\omega) = \frac{I_0}{2\pi}\left(\frac{T}{1 + i(\omega - \omega_0)T} + \frac{T}{1 - i(\omega - \omega_0)T}\right) = \frac{I_0 T}{\pi} \frac{1}{1 + \left[(\omega - \omega_0)T\right]^2} \, .$$

d) The maximum is $W(\omega_0) = I_0 T/\pi$ at the center frequency

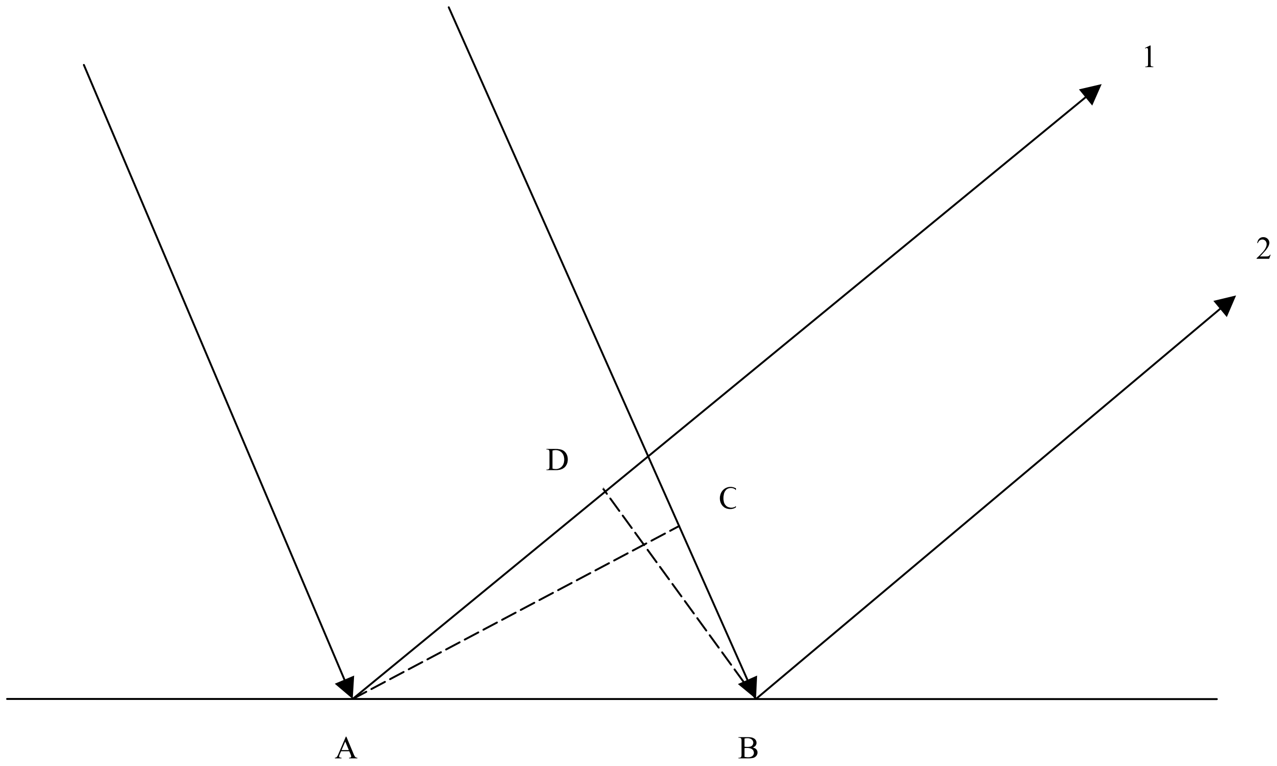
$$\omega = \omega_0 = 3.77 \cdot 10^{15} \text{ rad/s,}$$

and the half-width around this maximum is

$$2/T = 1.2 \cdot 10^{11} \text{ rad/s.}$$

(the value is reduced by a factor 1/2 relative to the maximum value at the frequencies $\omega = \omega_0 \pm 1/T$).

Problem 3



a) The rays are marked with arrows. The path-length difference between the two rays (1 and 2) reflected from two neighboring grooves at A and B is AD-BC, where the dotted lines AC and BD are normal to, respectively, the ray incident at B and the ray reflected at A. For the angles we therefore have: $\angle\text{BAC} = \theta_0$ and $\angle\text{ABD} = \theta$. With the grating constant a = AB, we therefore have for the path-length difference:
$s = \text{AD} - \text{BC} = a(\sin\theta - \sin\theta_0)$ QED!

b) The light in the two rays are in-phase if $s = m\lambda$ where $m = 0, \pm 1, \pm 2,..$ etc. and λ is the wavelength. Then the light reflected from any two groves are also in-phase, so that the reflected waves from all the groves interfere constructively. Therefore the diffraction orders are in the directions $\theta = \theta_m$ where θ_m is given by the *grating equation:*

$$a(\sin\theta_m - \sin\theta_0) = m\lambda;\, m = 0,\pm 1,\pm 2,...\text{etc.}$$

c) For the order at the height h= 0, we have

$$m_{max} = \frac{a}{\lambda}[\sin(\pi/2) - \sin\theta_0] = \frac{a}{\lambda}[1 - \sin(\arctan(500/30))],$$

and for the order closest to $h = 3h_0$ we have

$$m_{\min} = \frac{a}{\lambda}\left[\sin(\arctan(500/90)) - \sin(\arctan(500/30))\right].$$

The difference is

$$m_{max} - m_{\min} = \frac{a}{\lambda}\left[1 - \sin(\arctan(500/90))\right] = 24.987$$

We therefore have 24 diffraction orders with a height < $3h_0$.

d) We now have $U(x,y,0) = At(x) = \frac{A}{2}\left[1+\cos(2\pi x/a)\right]$ and obtain for the Fourier spectrum at $z = 0$:

$$A(u,v,0) = 2\pi A T(u)\delta(v),$$

where

$T(u) = \mathcal{F}\ \{t(x)\} = 2\pi A\left[\delta(u) + \tfrac{1}{2}\{\delta(u - 2\pi/a) + \delta(u - 2\pi/a)\}\right]$

because $\mathcal{F}\ \{1\} = 2\pi\delta(u)$ and $\mathcal{F}\ \{\cos(\alpha x)\} = \pi\left[\delta(u-\alpha) + \delta(u+\alpha)\right]$ (see Øving 12 and Eq.(1.3) in the lecture notes). The angular spectrum representation of the diffracted field (Eq. (2.9) in the notes) then yields:

$$U(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v,0)\exp\left[i\left(ux + vy + z\sqrt{k^2 - u^2 - v^2}\right)\right]dudv$$

$$= A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\delta(u) + \tfrac{1}{2}\{\delta(u - 2\pi/a) + \delta(u + 2\pi/a)\}\right]\delta(v)\exp\left[i\left(ux + vy + z\sqrt{k^2 - u^2 - v^2}\right)\right]dudv.$$

Here we have contributions for $v = 0$ and $u = 0, \pm 2\pi/a$, which give rise to only three plane waves:

$$U(x,y,z) = A\exp(ikz) + \tfrac{1}{2}A\exp\left[i\left(\frac{2\pi x}{a} + z\sqrt{k^2 - \left(\frac{2\pi}{a}\right)^2}\right)\right] + \tfrac{1}{2}A\exp\left[i\left(-\frac{2\pi x}{a} + z\sqrt{k^2 - \left(\frac{2\pi}{a}\right)^2}\right)\right]$$

The first term is a plane wave in the z direction, the two last terms are plane waves propagating at angles $\pm\theta$ with the z axis. Since $A\exp\left[ik\left(x\sin\theta + z\cos\theta\right)\right]$ describes a plane wave at an angle θ with the axis, we see that $\sin\theta = \frac{2\pi}{ka} = \frac{\lambda}{a}$. QED!

For $a < \lambda$, the factor $\sqrt{k^2 - \left(\frac{2\pi}{a}\right)^2} = 2\pi\sqrt{1/\lambda^2 - 1/a^2} = \frac{2\pi i}{a}\sqrt{1-(a/\lambda)^2}$ becomes purely imaginary, so that the two last plane waves are exponentially damped in the z-direction.