TFY4195/FY3100: Optics

Exam, May 14, 2004: Suggested solution

Problem 1

a) System matrix:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & f_1+f_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \underline{\begin{pmatrix} -\frac{f_2}{f_1} & f_1+f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}}.$$

Transfer matrix:

$$\begin{pmatrix} 1 & d' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{f_2}{f_1} & f_1+f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \underline{\begin{pmatrix} -\frac{f_2}{f_1} & f_1+f_2-\frac{f_2}{f_1}d-\frac{f_1}{f_2}d' \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}}.$$

For conjugate planes we have $B' = 0$, and the transfer matrix reduces to the imaging matrix:

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ -P & 1/\beta \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & 0 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix},$$

where β is the magnification and P is the power.

b) From B'=0 we obtain the image relation: $d' = \frac{f_2}{f_1}(f_1+f_2) - \left(\frac{f_2}{f_1}\right)^2 d$.

The magnification is: $\beta = A' = -f_2/f_1$ = -1/6.

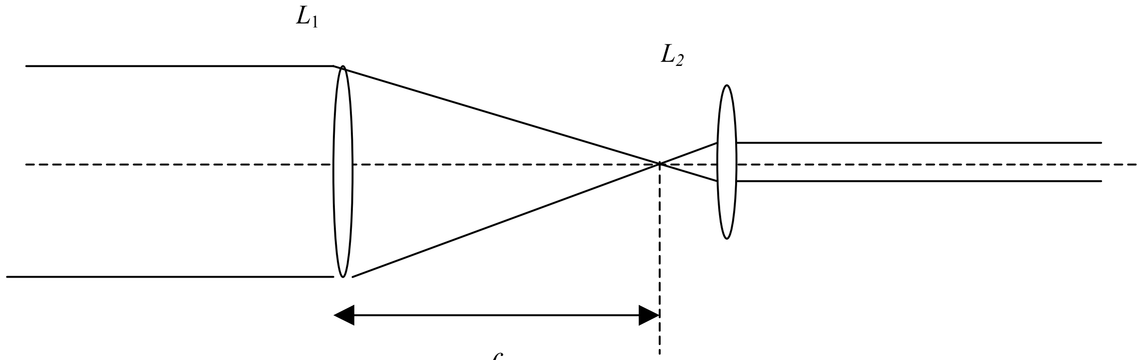
The principal planes H and H' are conjugate planes (images of each other) for which the magnification is: $\underline{\beta = +1}$. Here the magnification is $\beta = -f_2/f_1$ = -1/6 for *all* conjugate planes. Therefore <u>this system does not have principal planes.</u>

We here have P = 0, so this is a so-called <u>afocal system</u> (a telescope).

c)

- The <u>aperture stop</u> is the physical stop that limits the ray bundle from an object point on axis to the corresponding image point.
- The <u>entrance pupil</u> is the aperture stop imaged to the object space (i.e., seen from the object side).
- The <u>exit pupil</u> is the aperture stop imaged to the image space (i.e., seen from the image side).

- A chief ray is the ray from an object point to the corresponding image point that passes through the center of the aperture stop (and the associated pupils).
- The field stop is the stop that limits the bundle of chief rays from the object to the image.
- The entrance window is the field stop imaged to the object space (i.e., seen from the object side).
- The exit window is the field stop imaged to the image space (i.e., seen from the image side).



For the ray bundle from an object point on-axis we now have the situation illustrated in the figure above, and we immediately see that L_1 is the aperture stop and the entrance pupil. Since we only have two stops in the system, L_2 is the field stop and the exit window.

The exit pupil is L_1 imaged into image space by L_2. It is located at the image distance d_{EP}' where $\frac{1}{f_1+f_2}+\frac{1}{d_{EP}'}=\frac{1}{f_2}$, which yields: $d_{EP}'=\frac{f_2}{f_1}(f_1+f_2)$=5.83 cm to the right of L_2 and has the diameter $D_{EP}'=D_1\frac{d_{EP}'}{f_1+f_2}=D_1\frac{f_2}{f_1}$ =1.333 cm.

The entrance window is L_2 imaged into object space by L_1. It is located at the object distance d_{EW} where $\frac{1}{f_1+f_2}+\frac{1}{d_{EW}}=\frac{1}{f_1}$, which yields: $d_{EW}=\frac{f_1}{f_2}(f_1+f_2)$= 2.1 m to the left of L_1 and its diameter is $D_{EW}=D_2\frac{d_{EW}}{f_1+f_2}=D_2\frac{f_1}{f_2}$ =12 cm.

d) This is a telescope. The observer looks at a virtual image with image heigth $y'=\beta y=-\frac{f_2}{f_1}y=-y/6$. This image is both demagnified and inverted but it is observed from a distance that is much less than the original object distance. From the image relation in b) we see that the distance from the exit pupil to the image is:

$$d'-d_{EP}'=-\left(\frac{f_2}{f_1}\right)^2 d=-\beta^2 d\,.$$

The angle subtended by the object at the entrance pupil is *y*/*d,* but the corresponding angle subtended by the image at the exit pupil is $\left|\frac{y'}{d'-d_{EP}'}\right| = -\frac{1}{\beta}\frac{y}{d}$. So although the magnification is β= -1/6, the <u>visual magnification is:</u> $1/\beta = -f_1/f_2$ = -6.

Problem 2

a) The coherence function is:

$$\Gamma(\tau) = I_0 \int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} \exp(-i\omega\tau)d\omega = I_0 \frac{\exp[-i(\omega_0+\Delta\omega/2)\tau] - \exp[-i(\omega_0-\Delta\omega/2)\tau]}{-i\tau}$$

$$= I_0 \exp(-i\omega_0\tau)\frac{\exp(-i\Delta\omega\tau/2) - \exp(i\Delta\omega\tau/2)}{-i\tau}$$

$$= I_0\Delta\omega\exp(-i\omega_0\tau)\frac{\sin(\Delta\omega\tau/2)}{\Delta\omega\tau/2} = \underline{I_0\Delta\omega\exp(-i\omega_0\tau)\mathrm{sinc}(\Delta\omega\tau/2)}.$$

Substitution into the given formula yields:

$$I(s) = 2I_0\Delta\omega\left[1+\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\cos(\omega_0\, s/c)\right]$$

and we see that the interference term is the product $\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\cos(\omega_0\, s/c)$ where the sinc-term describes the envelope function. QED!

b) The visibility is defined as $V \equiv \underline{\frac{I_{max}-I_{\min}}{I_{max}+I_{\min}}}$, where I_{max} and I_{min} are, respectively, the maximum and minimum intensity as we move from one fringe to the next by changing the path-length difference.

In the interference signal $I(s) = 2I_0\Delta\omega\left[1+\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\cos(\omega_0\, s/c)\right]$, the cosine term describes the interference fringes while the sinc-term is the slowly varying envelope that determines the visibility. We then have

$I_{max} = 2I_0\Delta\omega\left[1+\left|\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\right|\right]$ and $I_{mim} = 2I_0\Delta\omega\left[1-\left|\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\right|\right]$, from which we obtain the visibility function:

$$V(s) = \frac{I_{max}-I_{\min}}{I_{max}+I_{\min}} = \underline{\left|\mathrm{sinc}\left(\frac{\Delta\omega\, s}{2c}\right)\right|}.$$

The first zero of the visibility function is at $\frac{\Delta\omega\, s}{2c} = \pi$, which yields for the longitudinal coherence length: $l_c = \underline{2\pi c / \Delta\omega}$.

c) When we go from one fringe to the next, s is changed by the amount: $2\pi c / \omega_0 = \lambda_0$. The number of interference fringes that can be observed with good visibility in the range $-l_c/2 \le s \le l_c/2$ is therefore: $l_c/\lambda_0 = \underline{\omega_0/\Delta\omega}$.

d) From the given data we obtain: $l_c = 20\lambda_0 = \underline{10\ \mu\text{m}}$, $\omega_0 = 2\pi c/\lambda_0 = \underline{3.77 \cdot 10^{15}\ \text{s}^{-1}}$, and $\Delta\omega = \omega_0 / 20 = 6\pi \cdot 10^{13}\ \text{s}^{-1} = \underline{18.85 \cdot 10^{13}\ \text{s}^{-1}}$.

Problem 3

a) From the given formulas the Fourier series expansion is given by:

$$t(y) = \sum_{m=-\infty}^{\infty} \frac{1}{a} T_0\left(m\frac{2\pi}{a}\right) \exp\left(im\frac{2\pi}{a}y\right),$$

$$T_0(v) = \mathcal{F}\{t_0(y)\} = \int_{-a/2}^{a/2} t(y)\exp(-ivy)dy.$$

We here have

$$T_0(v) = \int_{-d/2}^{d/2} \exp(-ivy)dy = d\frac{\sin(vd/2)}{vd/2} = d\operatorname{sinc}(vd/2)$$

and direct substitution yields the given formula. QED!

b) The field immediately behind the object can now be written as:

$$U(x,y,0) = A_0 t(y) = \sum_{m=-\infty}^{\infty} A_0 \frac{d}{a} \operatorname{sinc}(\pi md/a)\exp(2\pi imy/a).$$

A plane wave with amplitude A is given by:

$$A\exp(i\mathbf{k}\cdot\mathbf{r}) = A\exp\left[i\left(ux + vy + z\sqrt{k^2 - u^2 - v^2}\right)\right]$$

For $z = 0$ it reduces to: $A\exp[i(ux + vy)]$. Each term in the sum above has precisely this form: $A_m \exp[i(u_m x + v_m y)]$ with $A_m = A_0 \frac{d}{a}\operatorname{sinc}(\pi md/a)$, $u_m = 0$, and $v_m = 2\pi m/a$.

For $z \ge 0$ the m-th term in the Fourier series therefore gives rise to the plane wave:

$$A_m \exp(i\mathbf{k}_m \cdot \mathbf{r}) = A_0 \frac{d}{a}\operatorname{sinc}(\pi md/a)\exp\left[i\left(2\pi my/a + z\sqrt{k^2 - (2\pi m/a)^2}\right)\right]$$

c) The m-th plane wave propagates at the angle θ_m with the axis. From the given formula we then have

$$A_m \exp\left[i\left(2\pi my/a + z\sqrt{k^2 - (2\pi m/a)^2}\right)\right] = A_m \exp[ik(y\sin\theta_m + z\cos\theta_m)]$$

and immediately see that $2\pi m / a = k \sin\theta_m$, which yields the well-known grating equation: $a \sin\theta_m = m\lambda$.

d) Only a finite number of plane waves will be freely propagated because, for $|m| > a/\lambda$, we have so-called, evanescent plane waves that are exponentially damped in the *z* direction. With the given parameters we can have freely propagated waves for $|m| \le a/\lambda = 20$, so that 41 plane waves may be freely propagated (one for m=0, 20 for m<0, and 20 for m>0). However, some of these plane waves may be absent ("missing orders") because the amplitude A_m vanishes. With the given parameters we have: $A_m = A_0 \frac{1}{2} \operatorname{sinc}(\pi m/2)$, which vanishes for all $m = \pm 2, \pm 4, \pm 6$,..etc.
In this case we only have <u>21 freely propagated plane waves.</u>