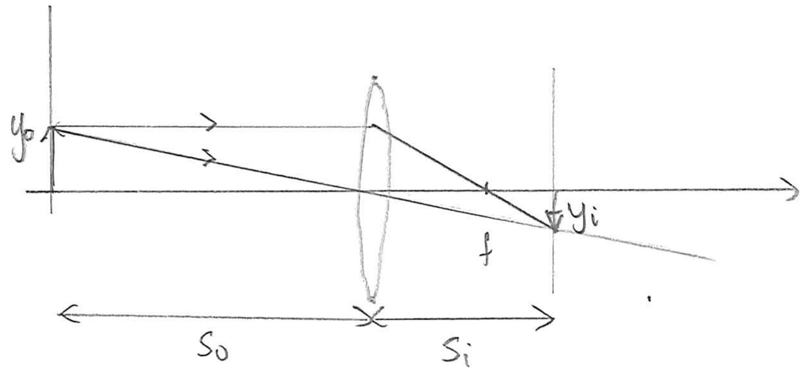
Solution TFY4195 Optics. Aug 2016

1a) See textbook.

b)



The image is formed in the plane where lines originating from the same point in the object plane meet.

$$M_T \equiv \frac{y_i}{y_o} \qquad \frac{-y_i}{s_i} = \frac{y_o}{s_o} \Rightarrow M_T = -\frac{s_i}{s_o}$$

$f = 50.0$ mm

$y_o = 1.70$ m

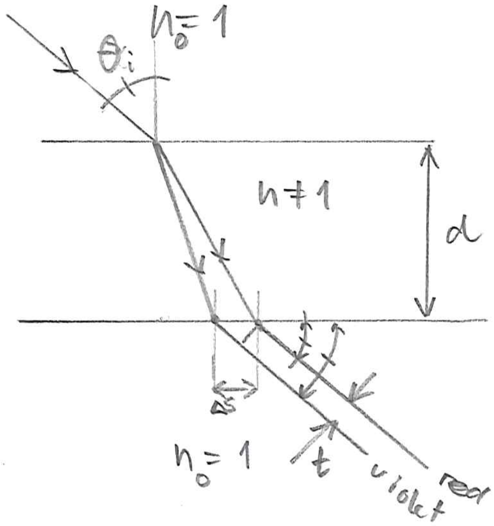
$s_o = 10.0$ m Eliminate s_i:

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \Rightarrow s_i = \frac{s_o f}{s_o - f}$$

$$M_T = -\frac{s_i}{s_o} = -\frac{f}{s_o - f} = \frac{f}{f - s_o} = -0.005$$

The image is demagnified and inverted.

1c)



$\alpha_i = 60°$
$d = 10.0\,\text{cm}$

$n_{red} = 1.505$ (red)
$n_{viol} = 1.545$ (violet)

$\theta_f = 60°$ for both, by symmetry.

Snell's law $\quad n_i \sin\theta_i = n_t \sin\theta_t$

$$\Delta S = S_{red} - S_{violet} = d(\tan\theta_{red} - \tan\theta_{viol})$$

$$\frac{t}{\Delta s} = \cos\theta_f = \frac{1}{2}$$

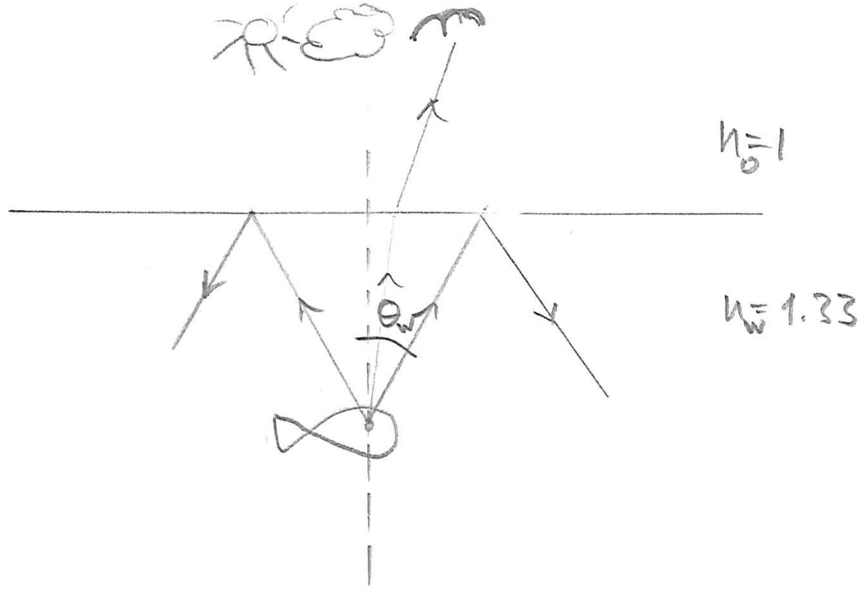
$$n_0 \sin\theta_i = n_{red}\sin\theta_{red} \Rightarrow \sin\theta_{red} = \frac{\sqrt{3}/2}{n_{red}}$$

$$\sin\theta_{viol} = \frac{\sqrt{3}/2}{n_{viol}}$$

$$t = \frac{1}{2}\Delta S = \frac{d}{2}\left(\tan\left(\arcsin\frac{\sqrt{3}/2}{n_{red}}\right) - \tan\left(\arcsin\frac{\sqrt{3}/2}{n_{viol}}\right)\right)$$

$$\underline{\underline{= 0.13\ \text{mm}}}$$

1d)



The explanation for this phenomenon is total reflection upon going from a dense to a less dense medium.

Snell's law $\quad n_w \sin\theta_w = \cancel{n_0}^{=1} \sin\theta_0$

Total reflection when $\theta_0 = \frac{\pi}{2} \Rightarrow \sin\theta_0 = 1$,

for $\theta_w = \theta_c$.

The critical angle is given by

$$n_w \sin\theta_c = 1$$

$$\theta_c = \arcsin\left(\frac{1}{n_w}\right) = \underline{48.6°}$$

The opening angle of the light cone reaching the fish is thus $\underline{\underline{2\theta_c = 97.2°}}$.

1e) $\vec{E}_1 = E_0\left(\hat{x} - \frac{2\pi}{3} e^{i\frac{\pi}{2}} \hat{y}\right) e^{ikz} e^{-i\omega t}$

This equation clearly describes a wave propagating in the z-direction with temporal dependence $e^{-i\omega t}$.
The polarization vector is in the x-y plane, as it should (e.g. in vacuum).

Jones: $\vec{E}_1 = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = E_0 \begin{bmatrix} 1 \\ -\frac{2\pi}{3} e^{i\frac{\pi}{2}} \end{bmatrix} = E_0 \begin{bmatrix} 1 \\ -\frac{2\pi i}{3} \end{bmatrix}$

Right handed, elliptical.

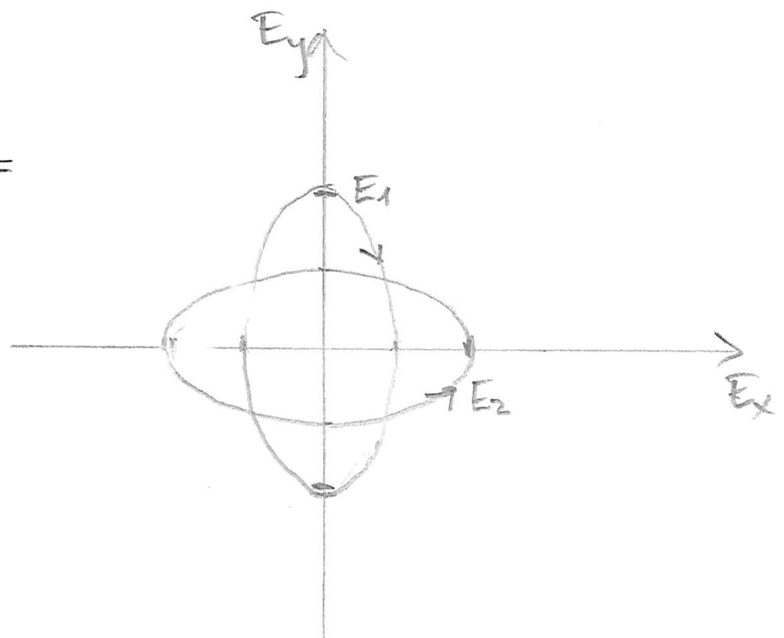
We note that the phase difference is $90°$, and the amplitude is larger in the y-direction by a factor $\underline{\frac{2\pi}{3} \approx 2}$

An orthogonal vector is given by $\vec{E}_1 \cdot \vec{E}_2^{\,*} = 0$

We see directly that

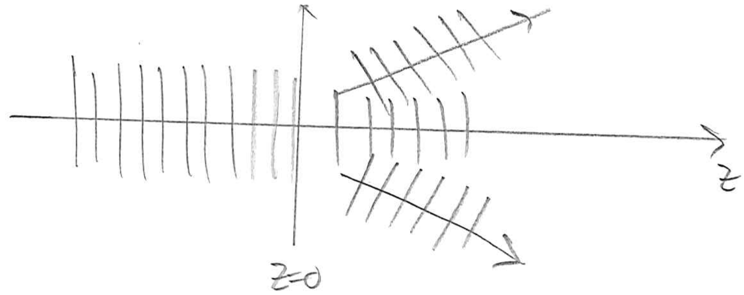
$\vec{E}_2^{\,*} = \begin{bmatrix} \frac{2\pi i}{3} \\ 1 \end{bmatrix}$ fulfills the orthogonality.

$$\underline{\underline{\vec{E}_2 = \begin{bmatrix} 2\pi/3 \\ 1 \end{bmatrix}}}$$



Task 2a) $t(x,y) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi y}{d}\right)$ @ $z=0$

Plane wave: $Ae^{ikz} = A\big|_{z=0}$



We note that there is no x-dependence in $t(x,y)$.
$\Rightarrow$ 1D problem.

$\cos u = \frac{1}{2}\left(e^{iu} + e^{-iu}\right)$, gives

$$t(x,y) = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\left(e^{i2\pi y/d} + e^{-i2\pi y/d}\right).$$

The field just after the object is $A\,t(x,y)$.
We see that it contains 3 plane waves, one with $k_y = 0$.
We must have $\vec{k}_i = \langle k_{x_i}, k_{y_i}, k_{z_i}\rangle$ and $|\vec{k}_i| = \frac{2\pi}{\lambda}$.

$U_i \sim e^{i\vec{k}_i\cdot\vec{r}}$

To find the k-vectors, we calculate the Fourier transform:

$$\mathcal{F}(t(y)) = \frac{1}{2}\int e^{ik_y y}\,dy + \frac{1}{4}\int e^{i(2\pi y/d + k_y y)}\,dy + \frac{1}{4}\int e^{i(-2\pi y/d + k_y y)}\,dy$$

$$= \frac{1}{2}\delta(k_y) + \frac{1}{4}\delta\left(k_y + \frac{2\pi}{d}\right) + \frac{1}{4}\delta\left(k_y - \frac{2\pi}{d}\right).$$

$\therefore$ 3 plane waves. // x-dep? No $\rightarrow \delta(k_x)$.

Zeroth order: $k_y = 0 \Rightarrow \vec{k} = \langle 0, 0, k_z\rangle = \langle 0, 0, k\rangle$

$\pm$ first order: $k_y = \pm\frac{2\pi}{d} \Rightarrow k_z = \sqrt{k^2 - \cancel{k_x^2} - \left(\pm\frac{2\pi}{d}\right)^2}$

2a conts).

$$U = U_0 + U_1 + U_{-1}$$

$$0: \quad U_0 = \frac{A}{2} e^{ikz}$$

$$\pm 1: \quad U_1 = \frac{A}{4} e^{i\langle k_x, k_y, k_z \rangle \cdot \langle x, y, z \rangle}$$

$$= \frac{A}{4} e^{i(k_y y + k_z z)} = \frac{A}{4} e^{i\left(\pm\frac{2\pi}{d} y + z\sqrt{k^2 - \left(\pm\frac{2\pi}{d}\right)^2}\right)}$$

To summarize: at $z > 0$ the propagated field can be calculated by the Fresnel diffraction integral (FDi).

For $z \gg 0$, the FDi simplifies to the Fourier transform.

By angular decomposition as done here, one can describe the field at $z > 0$ by exploiting conservation of momentum ($\vec{k}$) and adjusting each plane wave with the appropriate angle dependent phase term.

As $U = U(\vec{k})$, the intensity distribution at z can be calculated by taking an (inverse) FT of $U(\vec{k})$.

Task 2.

b) $$u = \frac{A}{2} e^{ikz} + \frac{A}{4} e^{i\left\{\pm \frac{2\pi y}{d} + z\sqrt{k^2 - \left(\frac{2\pi}{d}\right)^2}\right\}}$$

$$= \frac{A}{2} e^{ikz} + \frac{A}{4} e^{i\left\{\pm \frac{2\pi y}{d} + zk\sqrt{1 - \frac{\lambda^2}{d^2}}\right\}}$$

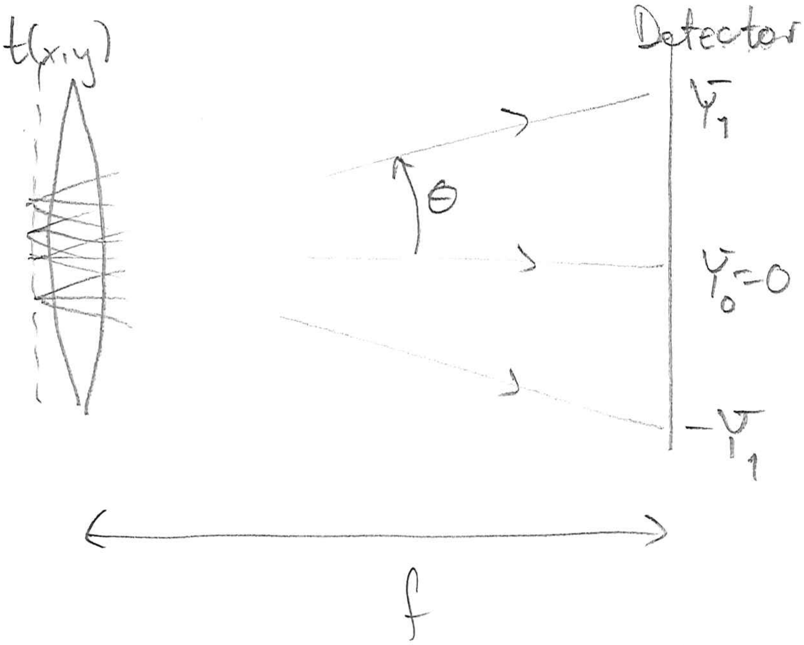
$$\simeq \frac{A}{2} e^{ikz} + \frac{A}{4} e^{i\left\{\pm \frac{2\pi y}{d} + zk - \frac{1}{\cancel{2}} \frac{\lambda^{\cancel{2}}}{d^2} \frac{\cancel{2}\pi z}{\cancel{\lambda}}\right\}}$$

$$= \frac{A}{2} e^{ikz} \left[1 + \frac{1}{2} e^{i\left\{\pm \frac{2\pi y}{d} - \frac{\pi \lambda z}{d^2}\right\}}\right]$$

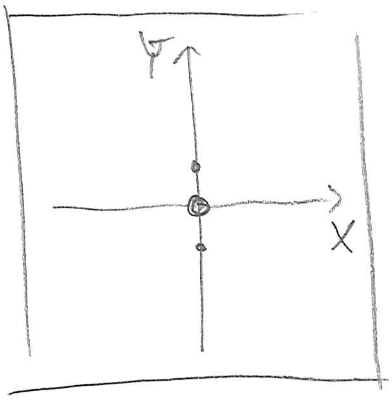
$$= \frac{A}{2} e^{ikz} \left[1 + \frac{1}{2} e^{-i\pi \frac{\lambda z}{d^2}} \left(e^{i\frac{2\pi y}{d}} + e^{-i\frac{2\pi y}{d}}\right)\right]$$

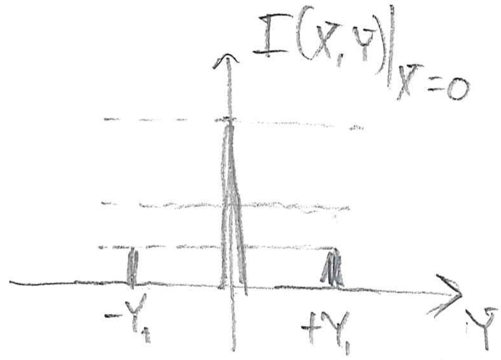
$$= \underbrace{\frac{A}{2} e^{ikz}}_{\text{plane wave}} \underbrace{\left[1 + e^{-i\pi \frac{\lambda z}{d^2}} \cos \frac{2\pi y}{d}\right]}_{\text{correction because of the periodic object.}}$$

2c)



- Central maximum is 4 times stronger than the two other.





From a): $\delta\left(k_y \pm \frac{2\pi}{d}\right)$

$$\tan\theta = \frac{Y}{f} \quad \& \quad \sin\theta = \frac{k_y}{k} \quad \& \quad k_y = \pm \frac{2\pi}{d}$$

- Assume small angles:

$$\Rightarrow \frac{Y}{f} \simeq \frac{k_y}{k} = \frac{\pm\frac{2\pi}{d}}{\frac{2\pi}{\lambda}} = \pm\frac{\lambda}{d}$$

$$\bar{Y}_{\pm 1} \simeq \pm f \frac{\lambda}{d} = \underline{\underline{\pm 5 \cdot 10^{-4}\,\text{m}}}$$

(Small angles, OK!)

2d) $U = U_0 + U_1 + U_{-1}$

$$= \frac{A}{2} e^{ikz} + \left\{ \frac{A}{4} e^{i\left(\pm 2\pi y/d + z\sqrt{k^2 - \left(\frac{2\pi}{d}\right)^2}\right)} \right\}_{U_1, U_{-1}}$$

$$= \frac{A}{2} e^{ikz} + \frac{A}{4} \left\{ e^{i\left(\pm 2\pi y/d + z\sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{2\pi}{d}\right)^2}\right)} \right\}_{U_1, U_{-1}} \qquad // \lambda \gg d$$

$$\simeq \frac{A}{2} e^{ikz} + \frac{A}{4} e^{i 2\pi y/d + iz\frac{2\pi}{d}} + \frac{A}{4} e^{-i 2\pi y/d + iz\frac{2\pi}{d}}$$

$$= \frac{A}{2} e^{ikz} + \frac{A}{4} e^{-2\pi z/d} \left(e^{i 2\pi y/d} + e^{-i 2\pi y/d} \right)$$

$$= \frac{A}{2} e^{ikz} + \cancel{\frac{A}{2} e^{-2\pi z/d}}^{\to 0} \cos \frac{2\pi y}{d} \longrightarrow \underline{\underline{\frac{A}{2} e^{ikz}}}$$

In this case, the amplitude is reduced to $\frac{A}{2}$.

With the exponential damping factor, there will be only a plane wave propagating in the z-direction.

Task 2e) If the grating is given a sideways translation such that $t'(x,y) = t(x,y-a)$, the intensity distribution will stay the same.

This can be shown to be a consequence of the Fourier shift theorem. $\tilde{\mathcal{F}}\{f(x-a)\} = e^{ika}\tilde{\mathcal{F}}\{f(x)\}$.

The scattering amplitude thus picks up an extra phase factor e^{ika}.

Because the intensity is the abs. square of the amplitude, the intensity is not changed:

$$I(x,y) = |\tilde{\mathcal{F}}\{t(x,y)\}|^2$$

$$\begin{aligned} I'(x,y) &= |\tilde{\mathcal{F}}\{t(x,y-a)\}|^2 \\ &= |e^{ika}\tilde{\mathcal{F}}\{t(x,y)\}|^2 \\ &= I(x,y). \end{aligned}$$

q.e.d.