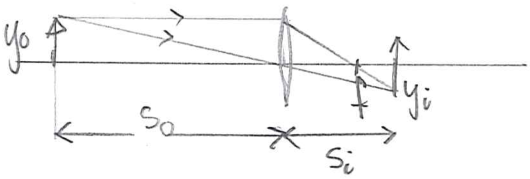
Solution. Exam TFY 4195 Dec. 2018

Task 1 a) $s_0 = 90$ cm

$s_i = 45$ cm

Focal length f is given by $\frac{1}{s_0} + \frac{1}{s_i} = \frac{1}{f}$ (Gauss)

$$\Rightarrow f = \left(\frac{1}{s_0} + \frac{1}{s_i}\right)^{-1} = \frac{s_0 s_i}{s_0 + s_i} = \frac{90 \cdot 45}{90+45} \text{ cm} = \underline{\underline{30 \text{ cm}}}$$



$$M_T \equiv \frac{y_i}{y_0} = -\frac{s_i}{s_0}$$

$$M_T = \frac{-45}{90} = \underline{\underline{-\frac{1}{2}}}$$

The image is <u>demagnified and inverted</u>.

Task 1b) The critical angle can be calculated from Snells law: $n_a \sin\theta_a = n_o \sin\theta_o$, with $\theta_o = \theta_c$ and $\theta_a = 90°$.

$$\Rightarrow n_a = n_o \sin\theta_c$$

$$\theta_c = \arcsin \frac{n_a}{n_o}.$$ Must have $n_a < n_o$ for total reflection.

Here, $\theta_c = \arcsin\left(\frac{1.00}{1.48}\right) = \underline{42.5°}$

$$n_w \sin\theta_i = n_o \sin\theta_c$$

$$\sin\theta_i = \frac{n_o}{n_w}\,\frac{n_a}{n_o}$$

$$\theta_i = \arcsin\left(\frac{n_a}{n_w}\right) = \underline{48.8°}$$

For incidence angles $\underline{\underline{\theta_i \geq 48.8°}}$, the beam will be reflected back into the water.

Task 1c)

A spherical wave is given by $\Psi = A\frac{e^{ikr}}{r}$

In Cartesian coordinates x, y, z, $r = \sqrt{x^2+y^2+z^2}$

Paraxial: $x, y \ll z$

$$r = z\sqrt{1+\frac{x^2+y^2}{z^2}} \approx z\left(1+\frac{x^2+y^2}{2z^2}\right) = z + \frac{x^2+y^2}{2z}.$$

As the phase varies faster than the denominator, one can often take $r \approx z$ in the denominator:

$$\Psi_{paraxial} = \frac{A}{z} e^{ik\left(z+\frac{x^2+y^2}{2z}\right)}$$

By using the Fresnel diffraction integral, assuming a point source at the origin, we get:

Source: $U_0(x,y,0) = A'\delta(x)\delta(y)$

$$U(x,y,z) = \frac{1}{i\lambda z} e^{ikz} e^{ik\frac{X^2+Y^2}{2z}} \underbrace{\iint U_0 \, e^{i\frac{k}{2z}(x^2+y^2) - i\frac{k}{z}(Xx+Yy)} \, dx dy}_{=A'}$$

$$= \frac{1}{i\lambda z} A' e^{ik\left(z+\frac{X^2+Y^2}{2z}\right)}$$

$\sim \Psi_{paraxial}$ when we note that X and Y are just the x, y coordinates at the observation plane, i is a phase constant ($i = e^{i\frac{\pi}{2}}$), and we can consider λ just a scaling factor.

Task 1 d) Given: $R_p = |r_p|^2 = \dfrac{\tan^2(\theta_i - \theta_t)}{\tan^2(\theta_i + \theta_t)}$.

At the Brewster angle, there is no reflection of p-polarized light. As $\tan(u) \xrightarrow[u \to 90°]{} \infty$, we note that $R_p = 0$ if $\theta_i + \theta_t = 90°$.

Snell's law reads $n_i \sin\theta_i = n_t \sin\theta_t$.

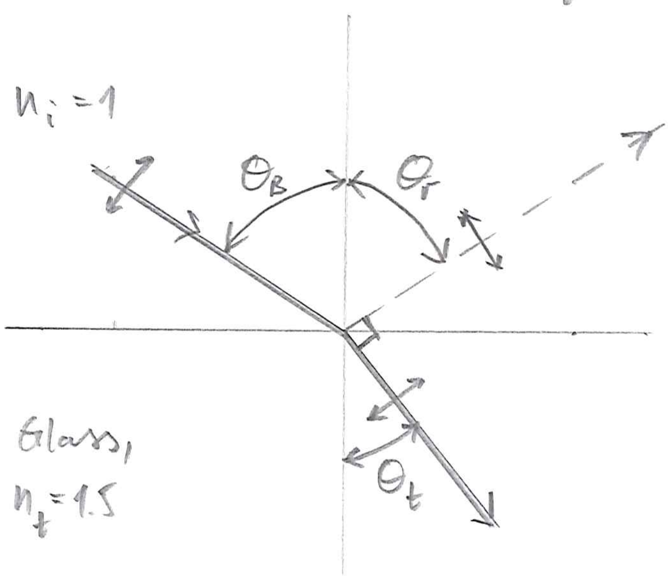
With $\theta_i = \theta_B$ we must have $\theta_t = 90° - \theta_i = 90° - \theta_B$:

$$n_i \sin\theta_B = n_t \sin(90° - \theta_B) = n_t \cos\theta_B$$

$$\underline{\underline{\tan\theta_B = \frac{n_t}{n_i}}} \qquad \text{(And the constant } \underline{C=1}\text{).}$$

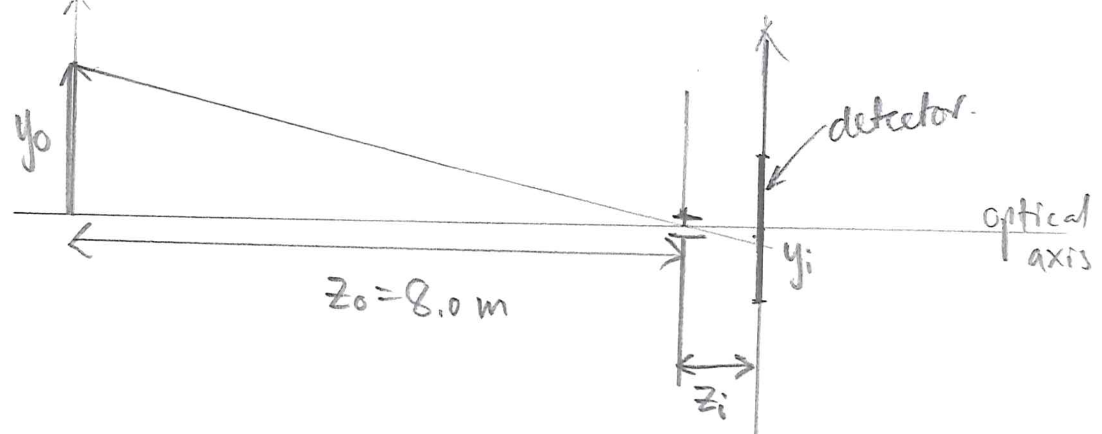
With $n_t = 1.5$ and $n_i = 1{,}0$, $\theta_B = \arctan(1.5) = \underline{56.3°}$

Thus, $\theta_r = \theta_B = 56{,}3°$ and $\theta_t = 90° - \theta_B = \underline{33.7°}$



We note that the dipole oscillations in the glass induced by the incoming field are parallel to the direction of the outgoing beam.
Parallel to dipole oscillations, there is no emitted field.
This simple argument supports that $R_p = 0$.

Task 2 a)



We see that $\dfrac{y_o}{z_o} = \dfrac{-y_i}{z_i}$

$$M_T = \frac{y_i}{y_o} = -\frac{z_i}{z_o} = -\frac{0.1\text{ m}}{8.0\text{ m}} = -0.0125$$

The image is demagnified and inverted.

The height of the image will be 0.025 m

The AS is the pinhole.

The EnP is the AS imaged to the object side ⇒ it is the pinhole

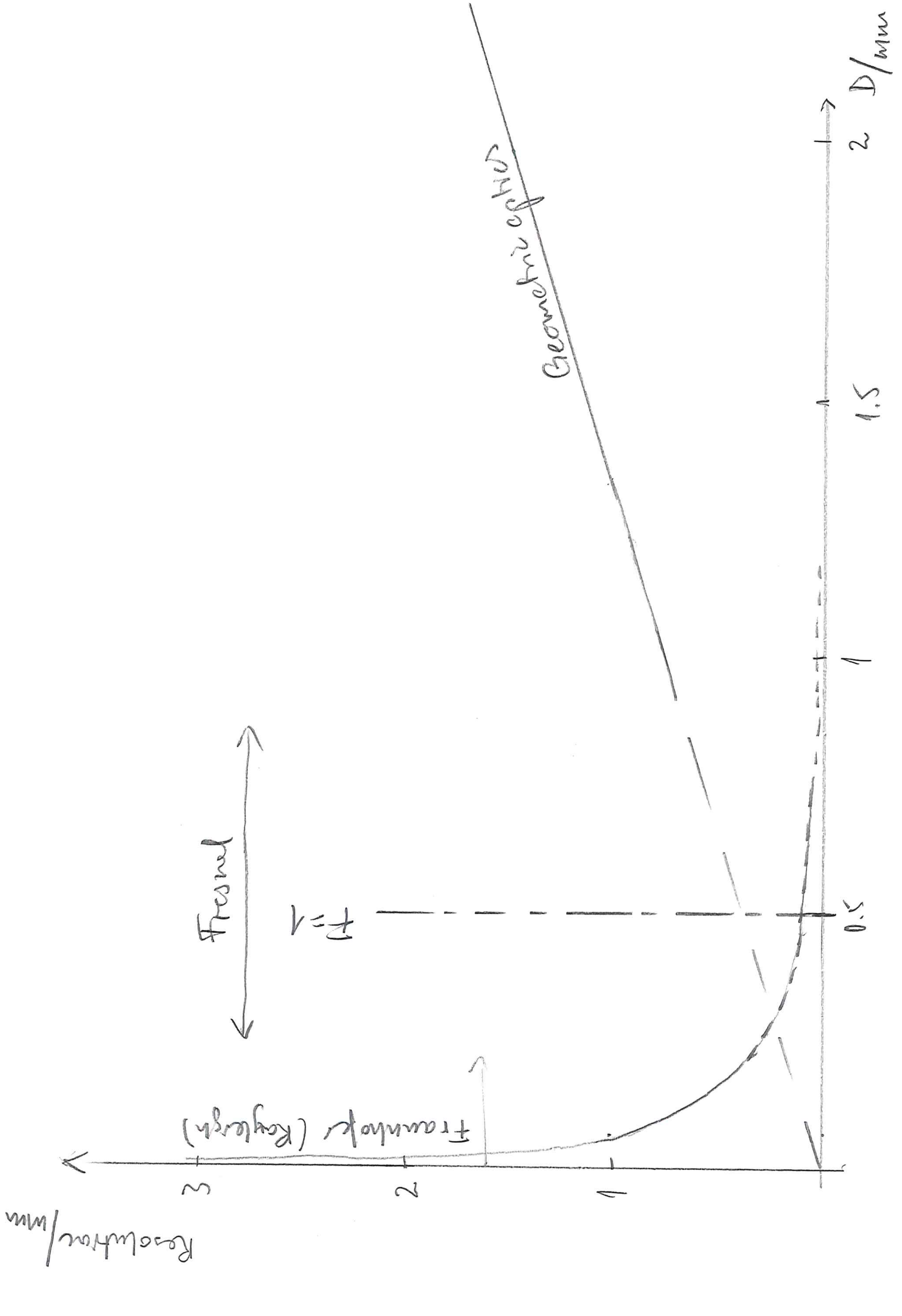
The ExP ——"—— image side ⇒ ——"——.

The field stop is detector.

The angular field of view is approximately

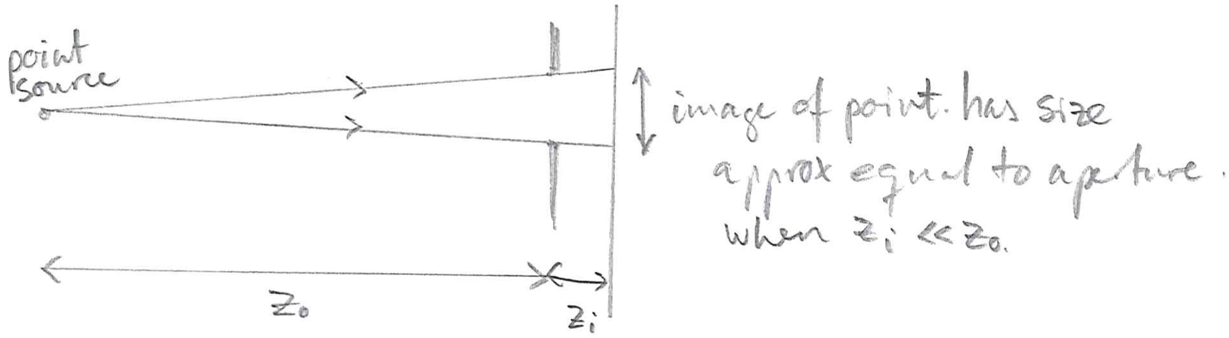
$$\alpha = 2\arctan\left(\frac{S_{det}/2}{z_i}\right) = 2\arctan\left(\frac{70/2}{100}\right) = 38.6^\circ$$

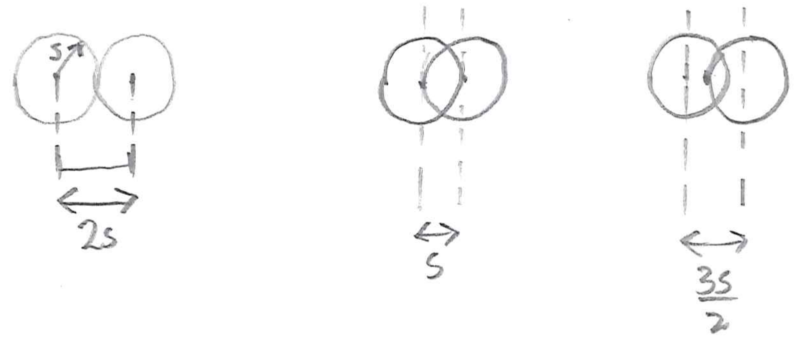
Task 2b, 2c, 2d



Geometric optics
Fresnel
F=1
Fraunhofer (Rayleigh)
D/mm
Resolution/mm
0.5
1
1.5
2
1
2
3

Task 2 b) In the geometric optics approximation, light moves in straight lines. Here $z_o \gg z_i$.





Arguably, a separation of $\sim \frac{3s}{2}$ should suffice to reliably distinguish the spots.

$$\frac{3s}{2} = \frac{3}{4}D.$$

Task 2c)

Fraunhofer regime. The Airy disk has radius $\rho_1 = 1.22 \frac{R\lambda}{2a}$,

where R is the distance from the pinhole to the screen $(= z_i)$

λ is given

a is the radius of the hole $\left(= \frac{D}{2}\right)$.

According to Rayleigh, two points are resolved if the center of the Airy disk for one of the points coincides with the 1st minimum of the other.

The resolution is thus equal to ρ_1, with

$$\rho_1 = 1.22 \frac{z_i \lambda}{D}$$ which is clearly a hyperbola.

$$\text{"}\rho_1 [mm] \approx 1.22 \cdot 100 \cdot 550 \cdot 10^{-6} \cdot \frac{1}{D[mm]}\text{"}$$

$$= 0.067 \frac{1}{D[mm]}.$$

The F-number is $F = \frac{(D/2)^2}{\lambda z_i}$.

$D = 2\sqrt{F \lambda z_i}$

$F = 1: D \simeq 0.5$ mm.

$F = 10^{-2}: D \simeq 0.05$ mm

$F = 10^{2}: D \simeq 5$ mm.

Task 2d

$$z_i = \frac{(D_{optim}/2)^2}{\lambda}$$

$$D_{optim} = 2\sqrt{z_i \lambda} = 2\sqrt{\lambda}\,\sqrt{z_i} = \underline{0.047\sqrt{z_i}}$$

Doptim corresponds to a Fresnel number

$$F \sim \frac{(D_{optim}/2)^2}{\lambda z_i} = 1$$

which is in the Fresnel regime.

With $z_i = 100$ mm, $\underline{\underline{D_{optim} \simeq 0.5 \text{ mm}}}$

If the linear dimensions of the camera are quadrupled, the diameter Doptim increases by a factor 2. // Doptim $\sim \sqrt{z_i}$.

In this case the field of view will be the same. The ratio detector size vs. resolution is doubled: The resolution will be improved <u>by a factor 2</u>.

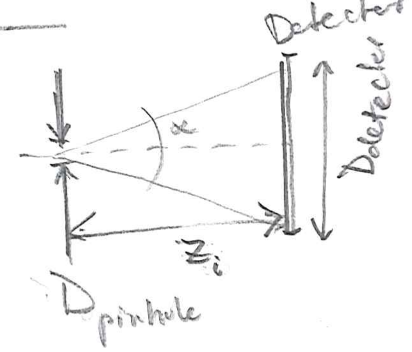
The <u>f-number</u> will be increased, i.e. the light-collecting properties will worsen.

To get the same exposure per detector area, ($\propto$ photons through pinhole, if you prefer), the exposure time must be quadrupled. This corresponds to an increase of the f-number by <u>2</u>

(see next page for illustrations).

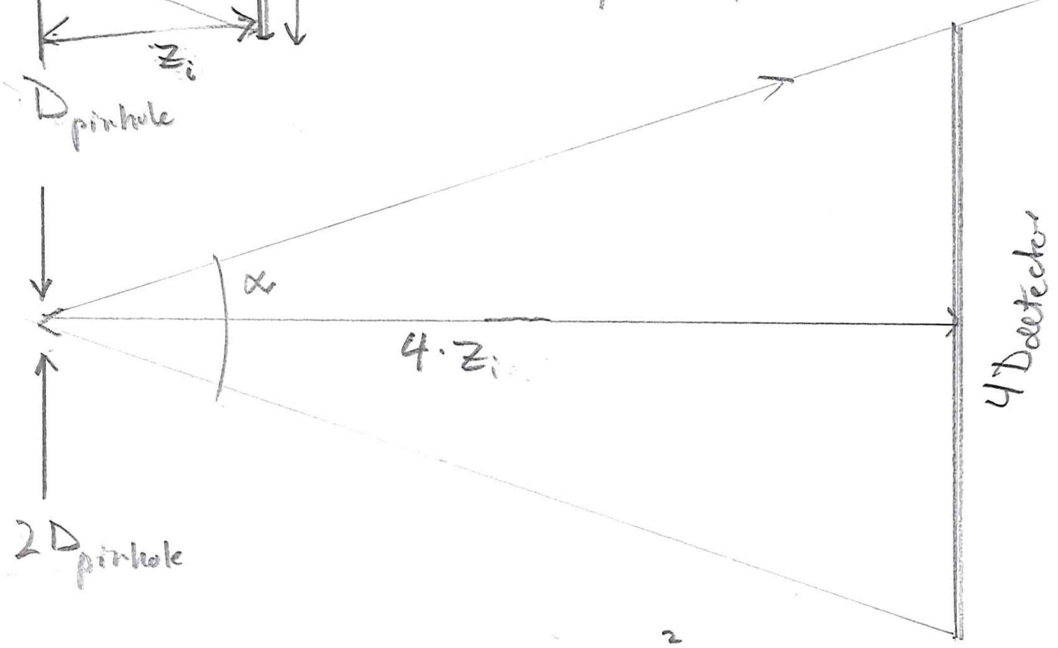
Task 2d cont's

Before:



α ~ field of view.

After:



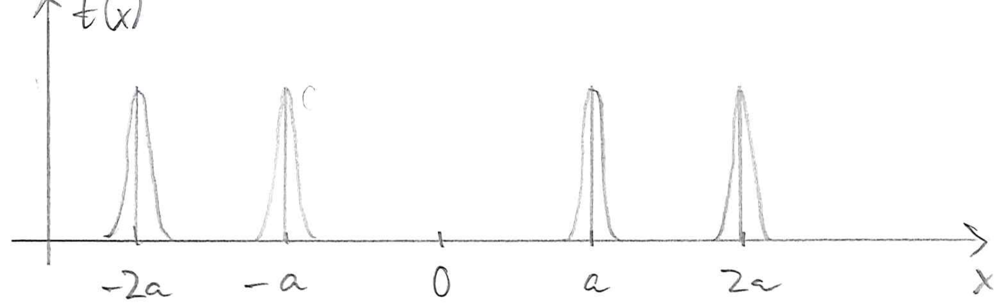
Photons per detector area $\sim \phi \dfrac{D_{pinhole}^2}{D_{detector}^2} = \rho$

$$\frac{\rho_{after}}{\rho_{before}} = \frac{\dfrac{(2D_{pinhole})^2}{(4D_{detector})^2}}{\dfrac{D_{pinhole}^2}{D_{detector}^2}} = \underline{\frac{1}{4}}$$

To get the same exposure per detector area, the exposure time must be increased by a factor $\underline{\underline{4}}$,

corresponding to an increase of the f-number by $\underline{\underline{2}}$.
(i.e. two stops à $\sqrt{2}$).

Task 3 a)



When $w \to 0$, we have 4 Dirac δ-functions.
We choose to put the origin in the position with the highest symmetry for simplest possible algebra.

$$t(x) = A \sum_{n \in \{-2,-1,1,2\}} \delta(x - na), \quad A \text{ is the strength (amplitude).}$$

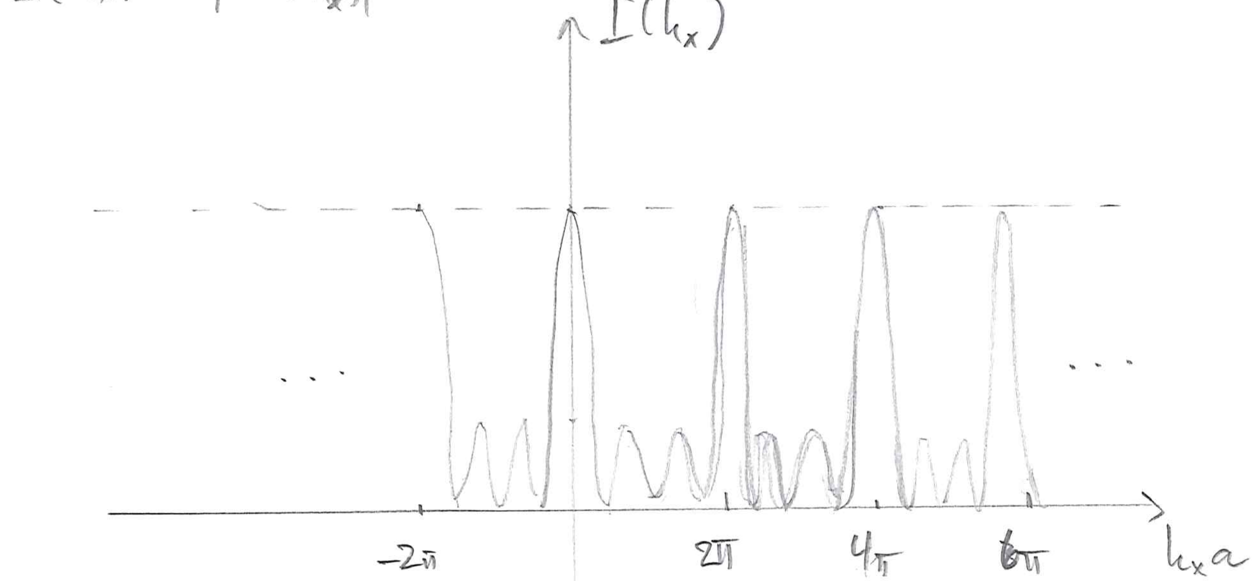
Fraunhofer diffraction $\Rightarrow$ $F(k_x)$ is F.T. of $t(x)$.

$$F(k_x) = \mathcal{F}\{t(x)\} = A \int \sum_n \delta(x-na) e^{ik_x x} dx = A \sum_n \int \delta(x-na) e^{ik_x x} dx$$

$$= A \sum_n e^{ik_x na} = A\left(e^{-2ik_x a} + e^{-ik_x a} + e^{ik_x a} + e^{2ik_x a}\right)$$

$$\underline{\underline{= 2A\left(\cos(2k_x a) + \cos(k_x a)\right)}} \sim \underline{\cos(2k_x a) + \cos(k_x a)}$$

$$I(k_x) \propto |F(k_x)|^2$$



Task 3b) Finite w. The easiest way to solve this task, is to note that $t_w(x)$ can be considered a convolution of 4 δ-functions with a Gaussian t_G,

$$t_w(x) = A \sum_{n=\{-2,-1,1,2\}} \delta(x-na) * e^{-x^2/w^2}$$

By the convolution theorem,

$$F_w(k_x) = \underbrace{\mathcal{F}\left\{A \sum \delta(x-na)\right\}}_{F(k_x),\text{ known from a)}} \cdot \mathcal{F}\left\{e^{-x^2/w^2}\right\}$$

We remember that the F.T. of a Gaussian is another Gaussian, or we can show it:

$$F_g(k_x) = \mathcal{F}\left\{e^{-x^2/w^2}\right\} = \int_{-\infty}^{\infty} e^{-x^2/w^2} e^{ik_x x} \, dx$$

Using the Gaussian integral given in the problem,

$$\underline{F_g(k_x) = w\sqrt{\pi}\, e^{-w^2k_x^2/4}}$$

We note that $F_g(k_x)$ gets narrower with increasing w.

Anyway,

$$I_w(k_x) \propto |F(k_x)|^2 \left|e^{-w^2k_x^2/4}\right|^2$$

$$= \underline{\underline{\left[\cos(2k_x a) + \cos(k_x a)\right]^2 e^{-w^2k_x^2/2}}} \qquad \text{qed.}$$