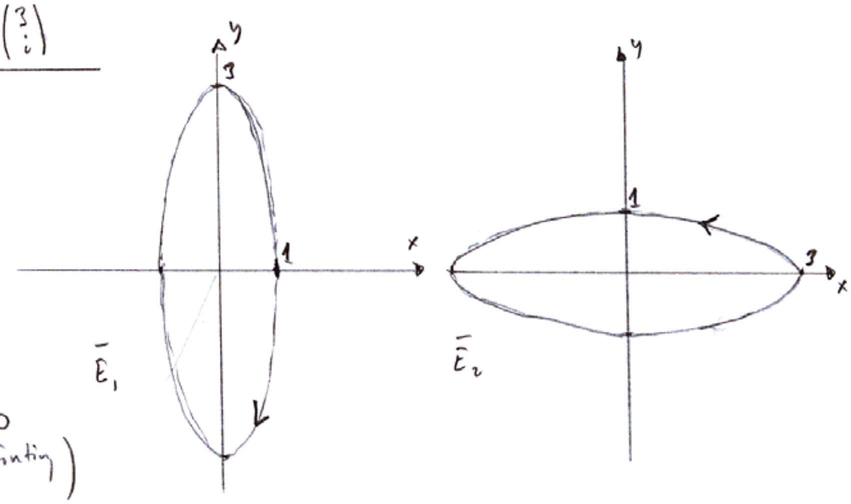
Optikk VK TFY4200
Solutions to examination June 3rd, 2004

P1: Ans: $\bar{E}_2 = \begin{pmatrix} 3 \\ i \end{pmatrix}$

Sketch:



$z = 0$
(z-axis pointing out)

Solution: $\bar{E}_1^* \cdot \bar{E}_2$ gives $1 \cdot e_{21} + (-3i)^* \cdot e_{22} = e_{21} + 3i\, e_{22} = 0$

$\Rightarrow e_{21} = 3 \;;\; e_{22} = i \;\Rightarrow\; \underline{\bar{E}_2 = \begin{pmatrix} 3 \\ i \end{pmatrix}}$

Sketch pol. ellipse

Def. $$\bar{J} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x}\, e^{i(kz-\omega t+\phi_x)} \\ E_{0y}\, e^{i(kz-\omega t+\phi_y)} \end{pmatrix} \Bigg|_{\substack{\phi_x=\phi_y=0 \\ z=0}} = \begin{pmatrix} E_{0x}\, e^{i(-\omega t)} \\ E_{0y}\, e^{i(-\omega t)} \end{pmatrix}$$

$$\bar{E}_1 = \begin{bmatrix} 1 \cdot e^{i(-\omega t)} \\ -3i \cdot e^{i(-\omega t)} \end{bmatrix} = \begin{pmatrix} 1 \cdot e^{i(-\omega t)} \\ 3\, e^{i(-\omega t - \frac{\pi}{2})} \end{pmatrix} \Bigg|_{\text{real part}} = \begin{pmatrix} 1 \cdot \cos(-\omega t) \\ 3 \cdot \cos(-\omega t - \frac{\pi}{2}) \end{pmatrix}$$ $-\frac{\pi}{2} - \frac{\pi}{2}$

in the same way

$$\bar{E}_2 = \begin{bmatrix} 3 \cdot \cos(-\omega t) \\ 1 \cdot \cos(-\omega t + \frac{\pi}{2}) \end{bmatrix}$$

Table

		$\omega t = 0$	$= \frac{\pi}{4}$	$= \frac{\pi}{2}$	$= \frac{3\pi}{4}$
$\bar{E}_1$	x	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
	y	0	$-3/\sqrt{2}$	-3	$3/\sqrt{2}$
$\bar{E}_2$	x	3	$3/\sqrt{2}$	0	$-3/\sqrt{2}$
	y	0	$-1/\sqrt{2}$	1	$1/\sqrt{2}$

P2:

Ans. $\frac{w_x}{w_y} \sim 0.44$ (0.4 - 0.5 range OK)

Detailed solution

At the focal plane the field distribution is the exact FT of the aperture field distribution.

$$U(x,y) = C \cdot \int_{-\frac{w_x}{2}}^{\frac{w_x}{2}} e^{-2\pi i \frac{x\xi}{\lambda z}} \cdot d\xi \int_{-\frac{w_y}{2}}^{\frac{w_y}{2}} e^{-2\pi i \frac{y\eta}{\lambda \cdot z}} d\eta$$

observation (FT) plane

aperture plane

Integrals are seperable. C complex constant.

Consider x-integral I_x: $\int_{-\frac{w_x}{2}}^{\frac{w_x}{2}} e^{-2\pi i \frac{x\xi}{\lambda z}} d\xi = \ldots = w_x \cdot \frac{\sin \pi\left(\frac{w_x x}{\lambda z}\right)}{\pi\left(\frac{w_x x}{\lambda z}\right)}$

I_x has zeros for $\frac{w_x x}{\lambda z} = m \cdot 1$ where $m \geq 1$

Hence, distance between minima inversely proportional to w_x $\left(x = \frac{m \cdot \lambda \cdot z}{w_x}\right)$. The same argument for vertical dimension (w_y & y).

$\frac{w_x}{w_y}$ can be directly measured in the figure.

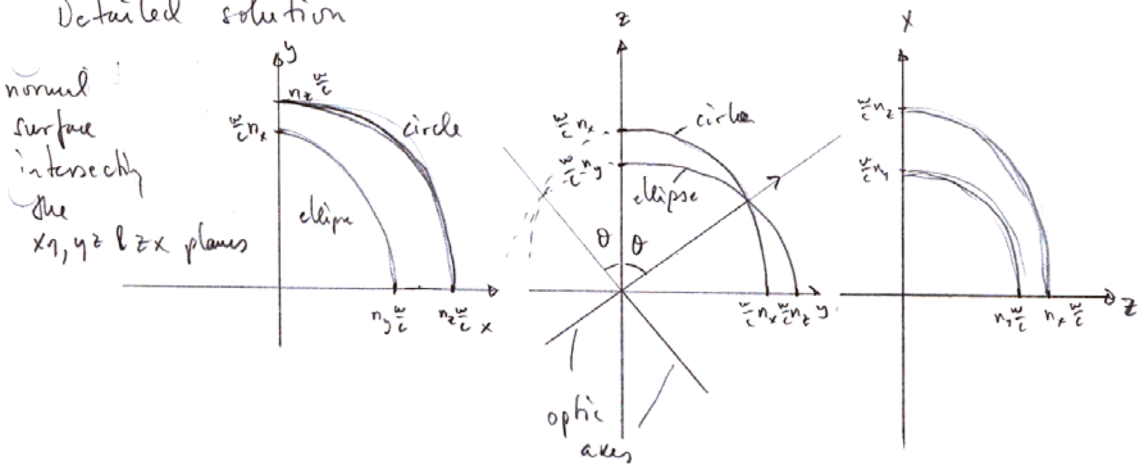
E.g. Locate the "seventh" minimum of each (x & y).

$\frac{w_x}{w_y}$ is $\frac{\frac{1}{x}[mm]}{\frac{1}{y}[mm]} = \frac{19.5 \text{ mm}}{44 \text{ mm}} \approx 0.443$

P3:

<u>Ans.</u> In the yz-plane ±54° from the z-axis.

Detailed solution



The normal surface(s) represent the solution(s) in terms of $\bar{k}$ or $\bar{v}$. It should be well known, easy to see that the optic axes are in the plane spanned by the extreme refractive indices (i.e. n_y & n_z → yz-plane)

In this plane the ordinary ray (circle) is independent of θ $n = n_x$. The extraordinary ray follows the usual

$$\frac{1}{n_{eff}^2} = \frac{\cos^2\theta}{n_y^2} + \frac{\sin^2\theta}{n_z^2}$$ with θ defined as above.

Along optic axis extraordinary and ordinary ray propagate with the same speed. ⇒

$$\frac{1}{n_x^2} = \frac{\cos^2\theta}{n_y^2} + \frac{\sin^2\theta}{n_z^2}$$; a little algebra gives $\theta = 54^\circ$

P4:

Ans.

	A	B	C
$\epsilon_{ijk}\epsilon_{irs}a_r b_k c_j d_s$	$c_i a_i b_j d_j - c_k d_k b_i a_i$	$(\bar{c}\times\bar{b})(\bar{a}\times\bar{d})$	$(\bar{c}\cdot\bar{a})(\bar{b}\cdot\bar{d})-(\bar{c}\cdot\bar{d})(\bar{b}\cdot\bar{a})$
$\epsilon_{ijk}a_i b_j c_k - c_i\epsilon_{ijk}a_j b_k$	$a_i\epsilon_{ijk}b_j c_k - c_k\epsilon_{kij}a_i b_j$	$\bar{a}\cdot(\bar{b}\times\bar{c})-\bar{c}\cdot(\bar{a}\times\bar{b})$	0
$\epsilon_{ijk}\epsilon_{rsi}a_j b_k c_s$	$a_j b_i c_i - a_k b_j c_k$	$\bar{c}\times(\bar{a}\times\bar{b})$	$(\bar{c}\cdot\bar{b})\bar{a}-(\bar{c}\cdot\bar{a})\bar{b}$

alternative answer — " —

	A	B	C
	3	3	2
	1	2	1
	2	1	3

Detailed solution.

row 1) First row trivial since only one combination have 4 ($\bar{a},\bar{b},\bar{c},\&\ \bar{d}$) in all cases!

row 2) $\epsilon_{ijk}a_i b_j c_k - c_i\epsilon_{ijk}a_j b_k = \epsilon_{ijk}a_i b_j c_k - \epsilon_{jki}a_j b_k c_i =$ [relabel indices of 2nd term $j\to i$, $k\to j$, $i\to k$

permutate cyclic $\epsilon_{jki}=\epsilon_{ijk}$

$= \epsilon_{ijk}a_i b_j c_k - \epsilon_{ijk}a_i b_j c_k = 0$

With similar arguments is shown that A1 =

$a_i\epsilon_{ijk}b_j c_k - c_k\epsilon_{kij}a_i b_j = 0$ C2

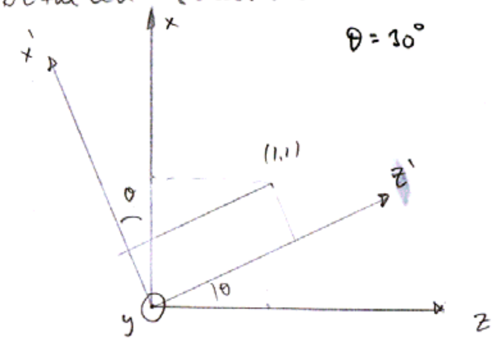
It should be known that $\bar{a}\cdot(\bar{b}\times\bar{c}) = \bar{c}\cdot(\bar{a}\times\bar{b})$ (B2)

row 3) What remains should be equal - indeed it is!

P5:

Ans.
$$\begin{pmatrix} 2.3275 & 0 & -0.1342 \\ 0 & 2.25 & 0 \\ -0.1342 & 0 & 2.4825 \end{pmatrix}$$

Detailed solution



we have that

$$z' = \frac{\sqrt{3}}{2} z + \frac{1}{2} x$$

$$x' = -\frac{1}{2} z + \frac{\sqrt{3}}{2} x$$

The point (1,1) in the zx-system becomes

$\left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2} \right)$ in the z'x' system,

seems OK.

Involving y the rotation matrix becomes

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}}_{\tilde{R} = l_{ij}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A 2nd rank tensor is transformed according to

$$a'_{ij} = l_{ip} l_{jq} \cdot a_{pq}$$

$a'_{11} = l_{11} l_{11} a_{11} + l_{11} l_{12} a_{12} + l_{11} l_{13} a_{13} + l_{12} l_{11} a_{21} + l_{12} l_{12} a_{22} + l_{12} l_{13} a_{23} + l_{13} l_{11} a_{31} + l_{13} l_{12} a_{32} + l_{13} l_{13} a_{33}$ and similar terms. Since a_{pq} diagonal and $l_{12} = l_{21} = l_{23} = l_{32} = 0$ it is easy to calculate a'_{ij}. Non-zero components are: $a'_{11} = \frac{3}{4} a_{11} + \frac{1}{4} a_{33}$; $a'_{13} = \frac{\sqrt{3}}{4} a_{11} - \frac{\sqrt{3}}{4} a_{33}$; $a'_{22} = a_{22}$; $a'_{31} = \frac{\sqrt{3}}{4} a_{11} - \frac{\sqrt{3}}{4} a_{33}$;

$a'_{33} = \frac{1}{4} a_{11} + \frac{3}{4} a_{33}$; insert tensor values gives answer.

P6:

Ans. $$r_{matrix} = \begin{array}{cl} \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ r_{31} & 0 & r_{33} \\ 0 & r_{42} & 0 \\ r_{51} & 0 & r_{53} \\ 0 & r_{62} & 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ yz \\ xz \\ xy \end{matrix} \end{array}$$

Detailed solution

follows from the definition of r_{matrix} and $s\tilde{y}$; see chapter 8 in compendium.

8: Detailed solution

A: SHG process $\omega \to 2\omega$

$\omega = \frac{2\pi c}{\lambda}$ => to produce $\lambda_{2\omega} = 488\,nm$ requires $\lambda_\omega : 976\,nm$

from graph: $\lambda_{2\omega} \to n_o = 1.680$ $n_e = 1.558$ (488 nm)

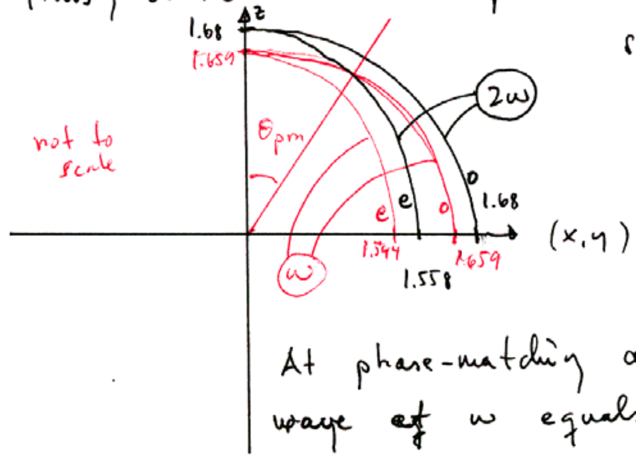
$\lambda_{\omega} \to n_o = 1.659$ $n_e = 1.544$ (976 nm)

Thus, oo→e collinear phase-matching is possible.

see graph

o - ordinary ray

e - extraordinary ray



At phase-matching angle (θ_{pm}) ref. index of ordinary wave of ω equals n_{eff} of e-wave @ 2ω

$$\Rightarrow \underset{\omega}{\underset{\uparrow}{\frac{1}{n_o^2}}} = \underset{2\omega}{\underset{\uparrow}{\frac{\cos^2\theta_{pm}}{n_o^2}}} + \underset{2\omega}{\underset{\uparrow}{\frac{\sin^2\theta_{pm}}{n_e^2}}}$$

values:

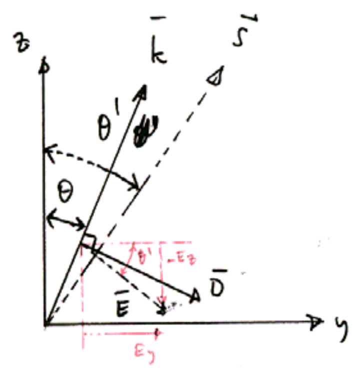
$$\cos\theta_{pm} = \sqrt{\frac{\frac{1}{1.659^2} - \frac{1}{1.558^2}}{\frac{1}{1.680^2} - \frac{1}{1.558^2}}} \Rightarrow \boxed{\theta_{pm} \approx 23.3^\circ}$$

Ans.

oo → e SHG possible

Detailed solution

(P7:) Take the coordinate system so that the optic axis is along z and the wave-vector $\bar{k}$ in the yz plane.



For the extraordinary ~~wave~~ beam the vector $\bar{B}$ (and $\bar{H}$) is perpendicular to the yz-plane.
The displacement field is always perpendicular to $\bar{k}$.

Thus,
$$\begin{cases} D_y = D\cos\theta \\ D_z = -D\sin\theta \end{cases}$$

The components of the electric field are

$$E_y = \frac{D_y}{\varepsilon_0 n_y^2} = \frac{D\cos\theta}{\varepsilon_0 n_y^2} = \frac{D\cos\theta}{\varepsilon_0 n_o^2}$$

$$E_z = \frac{D_z}{\varepsilon_0 n_z^2} = \frac{-D\sin\theta}{\varepsilon_0 n_z^2} = \frac{-D\sin\theta}{\varepsilon_0 n_e^2}$$

The direction of the ray vector is proportional to the Poynting vector $\bar{P} \propto (\bar{E}\times\bar{B})$ (or $(\bar{E}\times\bar{H})$).
Thus, the ray is in the yz-plane, but not along $\bar{k}$. The angle between $\bar{P}$ and the optic axis (z) is then

$$-\frac{E_z}{E_y} = \underline{\tan\theta'} = \frac{\cancel{D}\sin\theta\,\cancel{\varepsilon_0}n_o^2}{\cancel{\varepsilon_0}n_e^2\cdot\cancel{D}\cos\theta} = \underline{\frac{n_o^2}{n_e^2}\tan\theta}$$

answer

8:

B:

$$
\text{d-matrix} = \begin{array}{c} \\ x \\ y \\ z \end{array}
\begin{array}{c}
\begin{array}{cccccc} xx & yy & zz & yz & xz & xy \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & d_{15} & -d_{22} \\
-d_{22} & d_{22} & 0 & d_{15} & 0 & 0 \\
d_{31} & d_{31} & d_{33} & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Kleinman symmetri: we can permutate all x, y, z indices freely.

$\Rightarrow$ d_{15} : stems from χ_{xxz}) equal

$d_{24} (= d_{15})$: stems from χ_{yyz}) equal

d_{31} : stems from χ_{zxx}

$d_{32} (= d_{31})$: stems from χ_{zyy}) equal

$\Rightarrow$ all equal $= d_{15}$ (or d_{31})

o: $(-\sin\phi, \cos\phi, 0)$

e: $(\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta)$

project to form d_{eff} for oo→e case (SHG)

$$
\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 & d_{31} & -d_{22} \\
-d_{22} & d_{22} & 0 & d_{31} & 0 & 0 \\
d_{31} & d_{31} & d_{33} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} E_x^2 \\ E_y^2 \\ E_z^2 \\ 2E_yE_z \\ 2E_xE_z \\ 2E_xE_y \end{pmatrix}
= \cdots\cdots =
\begin{bmatrix} -2d_{22}E_xE_y \\ d_{22}(E_y^2 - E_x^2) \\ d_{31}(E_x^2 + E_y^2) \end{bmatrix} =
$$

2× o-wave

Ez-comp = 0

$$
= \begin{pmatrix} 2d_{22}\sin\phi\cos\phi \\ d_{22}(\cos^2\phi - \sin^2\phi) \\ +d_{31}\underbrace{\cancel{\quad}}_{=1} \end{pmatrix}
\quad \text{project onto e-wave (@ } 2\omega) \quad \cdot
\begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}
$$

$$= -d_{31}\sin\theta + \cos\theta\, d_{22}\left(3\sin\phi\cos^2\phi - \sin^3\phi\right)$$

$$= -d_{31}\sin\theta + d_{22}\cos\theta\sin 3\phi$$ ← Answer d_{eff} (oo→e)

other variants ok

e.g. $3\sin\phi - 4\sin^3\phi = \sin 3\phi$

$\theta = \theta_{pm}$ (8A)

ϕ chosen for optimum, e.g. $\phi = 30°$

9:

start with $\frac{1}{\mu_0} \cdot \left(\frac{n}{c}\right)^2 \bar{s} \times \bar{s} \times \bar{E} = -\bar{D}$

use $\bar{D} = \varepsilon_0 \cdot \tilde{K} \cdot \vec{E}$ & $\frac{1}{\varepsilon_0 \mu_0} = c^2$; rewrite in tensor notation

Note

$\Rightarrow n^2 \epsilon_{imn} \epsilon_{ijk} s_n s_j E_k = -K_{mq} \cdot E_q$

some cleaning using $(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})$

$\Rightarrow n^2 (E_m - s_m s_n \cdot E_n) = K_{mq} \cdot E_q$

$\bar{s}$ arbitrary ; K_{mq} taken as diagonal $\Rightarrow K_{mq} \cdot E_q = (n_m)^2 E_m$

(i.e. $= n_x^2 \cdot E_x + n_y^2 E_y + n_z^2 E_z$)

$\Rightarrow (n^2 - (n_m)^2)_m E_m = n^2 s_m \cdot s_n \cdot E_n$

or

write out !

$$E_m = \frac{n^2 s_m s_n \cdot E_n}{(n^2 - n_m^2)}$$

multiply with s_m on both sides $\Rightarrow$

$$\underbrace{\cancel{s_m E_m}}_{=1} = \frac{n^2 s_m^2 \cancel{s_n E_n}}{(n^2 - n_m^2)}$$

$s_m E_m$ and $s_n E_n$ scalar products cancel

$$\Rightarrow \frac{1}{n^2} = \frac{s_m^2}{(n^2 - n_m^2)} = \frac{s_x^2}{(n^2 - n_x^2)} + \frac{s_y^2}{(n^2 - n_y^2)} + \frac{s_z^2}{(n^2 - n_z^2)}$$

Q.E.D.

See problem 6.15 in compendium !!