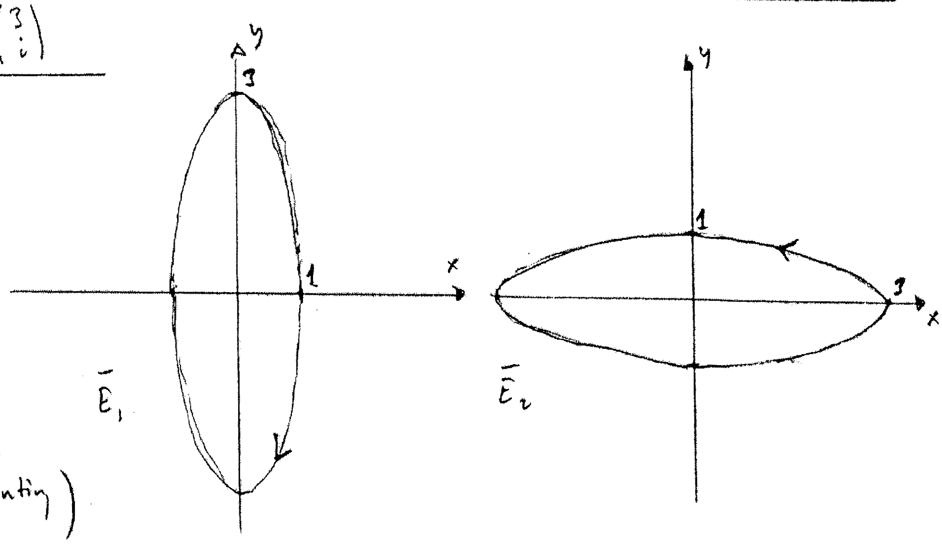
Optikk VK TFY4200

Solutions to examination June 3rd, 2004

P1: Ans: $\bar{E}_2 = \begin{pmatrix} 3 \\ i \end{pmatrix}$

Sketch:



$z = 0$
(z-axis pointing out)

Solution: $\bar{E}_1^* \cdot \bar{E}_2$ gives $1 \cdot e_{21} + (-3i)^* \cdot e_{22} = e_{21} + 3i e_{22} = 0$

$\Rightarrow e_{21} = 3$; $e_{22} = i$ $\Rightarrow$ $\underline{\bar{E}_2 = \begin{pmatrix} 3 \\ i \end{pmatrix}}$

Sketch pol. ellipse

Def. $$\bar{J} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x} e^{i(kz-\omega t+\phi_x)} \\ E_{0y} e^{i(kz-\omega t+\phi_y)} \end{pmatrix} \Bigg|_{\substack{\phi_x=\phi_y=0 \\ z=0}} = \begin{pmatrix} E_{0x} e^{i(-\omega t)} \\ E_{0y} e^{i(-\omega t)} \end{pmatrix}.$$

$$\bar{E}_1 = \begin{bmatrix} 1 \cdot e^{i(-\omega t)} \\ -3i \cdot e^{i(-\omega t)} \end{bmatrix} = \begin{pmatrix} 1 \cdot e^{i(-\omega t)} \\ 3 \, e^{i(-\omega t - \frac{\pi}{2})} \end{pmatrix} \Bigg|_{\text{real part}} = \begin{pmatrix} 1 \cdot \cos(-\omega t \\ 3 \cdot \cos(-\omega t - \frac{\pi}{2}) \end{pmatrix}$$

in the same way

$$\bar{E}_2 = \begin{bmatrix} 3 \cdot \cos(-\omega t) \\ 1 \cdot \cos(-\omega t + \frac{\pi}{2}) \end{bmatrix}$$

Table

		$\omega t = 0$	$= \frac{\pi}{4}$	$= \frac{\pi}{2}$	$= \frac{7\pi}{4}$
$\bar{E}_1$	x	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
	y	0	$-3/\sqrt{2}$	-1	$3/\sqrt{2}$
$\bar{E}_2$	x	3	$3/\sqrt{2}$	0	$-3/\sqrt{2}$
	y	0	$\cdot 1/\sqrt{2}$	1	$1/\sqrt{2}$

The Stokes vector is given by:

$$S_0 = \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle = 1+9 = 10$$

$$S_1 = \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle = -8$$

$$S_2 = 2\langle |E_x| \cdot |E_y| \rangle \cos\Delta = 2 \cdot 1 \cdot 3 \cdot \cos(-\pi/2)$$
$$= 0$$

$$S_3 = 2\langle |E_x||E_y| \rangle \sin\Delta = 2 \cdot 1 \cdot 3 \cdot \sin(-\pi/2)$$
$$= -6$$

P2:

Ans. $\frac{w_x}{w_y} \sim 0.44$ (0.4 - 0.5 range OK)

Detailed solution

At the focal plane the field distribution is the exact FT of the aperture field distribution.

$$\underbrace{U(x,y)}_{\text{observation (FT) plane}} = C \cdot \int_{-\frac{w_x}{2}}^{\frac{w_x}{2}} e^{-2\pi i \frac{x\xi}{\lambda z}} \cdot d\xi \int_{-\frac{w_y}{2}}^{\frac{w_y}{2}} e^{-2\pi i \frac{y\eta}{\lambda \cdot z}} d\eta$$

aperture plane

Integrals are seperable. C complex constant.

Consider x-integral I_x: $\int_{-\frac{w_x}{2}}^{\frac{w_x}{2}} e^{-2\pi i \frac{x\xi}{\lambda z}} d\xi = \dots = w_x \cdot \frac{\sin \pi\left(\frac{w_x x}{\lambda z}\right)}{\pi\left(\frac{w_x x}{\lambda z}\right)}$

I_x has zeros for $\frac{w_x x}{\lambda z} = m \cdot 1$ where $m \geq 1$

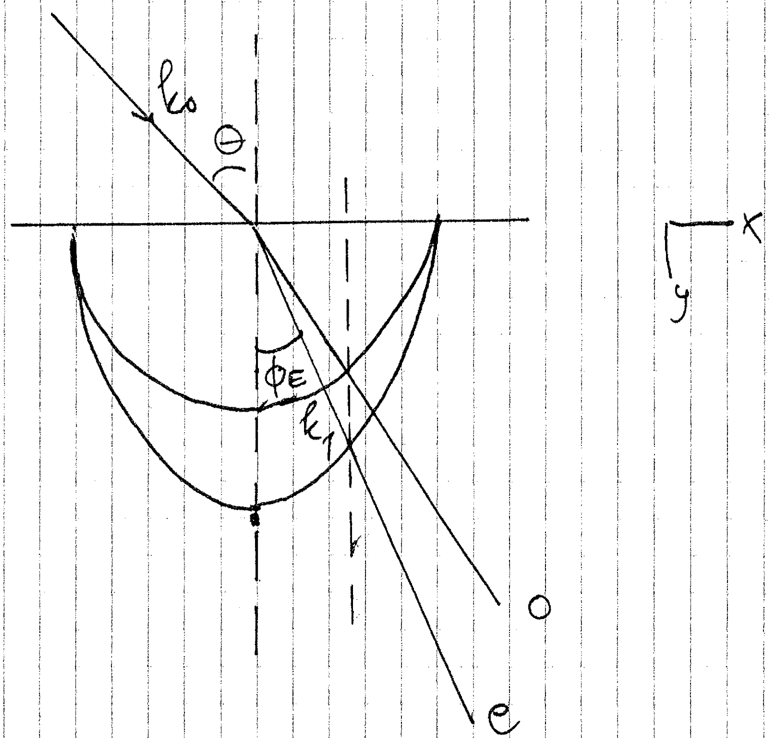
Hence, distance between minima inversely proportional to w_x $\left(x = \frac{m \cdot \lambda \cdot z}{w_x} \right)$. The same argument for vertical dimension (w_y & y).

$\frac{w_x}{w_y}$ can be directly measured in the figure.

E.g. Locate the "seventh" minimum of each (x & y).

$\frac{w_x}{w_y}$ is $\frac{\frac{1}{x}[mm]}{\frac{1}{y}[mm]} = \frac{19.5 \text{ mm}}{44 \text{ mm}} \approx 0.443$

Probl 3.



Momentum conservation along boundary

$$k_0 \sin\Theta = k_1 \sin\phi_E$$

Ellipse: $$\frac{k_x^2}{(n_o \frac{\omega}{c})^2} + \frac{k_y^2}{(n_E \frac{\omega}{c})^2} = 1$$

$$\Rightarrow \quad \frac{(\frac{\omega}{c}\sin\Theta)^2}{(n_o \frac{\omega}{c})^2} + \frac{k_y^2}{(n_o \frac{\omega}{c})^2} = 1$$

$$\Rightarrow \quad k_y = n_E \left(\frac{\omega}{c}\right) \sqrt{1 - \frac{\sin^2\Theta}{n_o^2}}$$

$$\mathrm{tg}\,\phi_E = \frac{k_x}{k_y} = \frac{n_o \sin\Theta}{n_E \sqrt{n_o^2 - \sin^2\Theta}}$$

4: The rotation matrix is given by

$$\bar{\bar{R}} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

With $\phi = -60°$ we get:

$$\bar{\bar{R}} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2}\sqrt{3} \\ 0 & 1 & 0 \\ -\frac{1}{2}\sqrt{3} & 0 & \frac{1}{2} \end{pmatrix}$$

And:

$$\bar{\bar{\epsilon}}' = \bar{\bar{R}}\,\bar{\bar{\epsilon}}\,\bar{\bar{R}}^{-1}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2}\sqrt{3} \\ 0 & 1 & 0 \\ -\frac{1}{2}\sqrt{3} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2.25 & 0 & 0 \\ 0 & 2.25 & 0 \\ 0 & 0 & 2.56 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2}\sqrt{3} \\ 0 & 1 & 0 \\ +\frac{1}{2}\sqrt{3} & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2.48\,25 & 0 & 0.1342 \\ 0 & 2.25 & 0 \\ 0.1342 & 0 & 2.3275 \end{pmatrix}$$

5:

Vi start with the

The "wavevectors" ; se Mikael's solution Ch. 7.

$$\hat{e}_o = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

$$\hat{e}_e = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} d_{31}(E_x^{\omega_1}E_z^{\omega_2} + E_z^{\omega_1}E_x^{\omega_2}) - d_{22}(E_x^{\omega_1}E_y^{\omega_2} + E_y^{\omega_1}E_x^{\omega_2}) \\ -d_{22}(E_x^{\omega_1}E_x^{\omega_2}) + d_{22}E_y^{\omega_1}E_y^{\omega_2} + d_{31}(E_y^{\omega_1}E_z^{\omega_2} + E_z^{\omega_1}E_y^{\omega_2}) \\ d_{31}(E_x^{\omega_1}E_x^{\omega_2} + E_y^{\omega_1}E_y^{\omega_2}) + d_{33}E_z^{\omega_1}E_z^{\omega_2} \end{pmatrix}$$

Insert E^{ω_1} and E^{ω_2} as e-waves $\Rightarrow$

$$= \begin{pmatrix} d_{31}(-2\cos\phi\sin\theta\cos\theta) - d_{22}(2\sin\phi\cos\phi\cos^2\theta \\ -d_{22}(\cos^2\phi\cos^2\theta - \sin^2\phi\cos^2\theta) + d_{31}(-2\sin\phi\sin\theta\cos\theta \\ d_{31}(\cos^2\phi\cos^2\theta + \sin^2\phi\cos^2\theta) + d_{33}\sin^2\theta \end{pmatrix}$$

Project out the o-wave $\Rightarrow$ eeo

$$d_{eff} = d_{31}(2\sin\phi\cos\phi\sin\theta\cos\theta) + d_{22}(2\sin^2\phi\cos\phi\cos^2\theta)$$
$$- d_{22}(\cos^3\phi\cos^2\theta - \sin^2\phi\cos\phi\cos^2\theta) + d_{31}(-2\sin\phi\cos\phi\sin\theta\cos\theta)$$

$$= d_{31}(0) - d_{22}(\cos^3\phi\cos^2\theta - 3\sin^2\phi\cos\phi\cos^2\theta)$$
$$= -d_{22}(4\cos^3\phi - 3\cos\phi)\cos^2\theta$$
$$= -d_{22}\cos^2\theta\cos 3\phi$$

eo:

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} d_{31}(\sin\theta\sin\phi - d_{22}(\cos^2\phi\cos\theta - \sin^2\phi\cos\theta) \\ -d_{22}(-\sin\phi\cos\phi\cos\theta) + d_{22}\sin\phi\cos\phi\cos\theta + d_{31}(-\sin\theta\cos\phi \\ \underbrace{d_{31}(-\cos\phi\sin\phi\cos\theta + \sin\phi\cos\phi\cos\theta)}_{=0} \quad d_{33}\cdot 0 \end{pmatrix}$$

Project out the e-wave

$$\Rightarrow d_{31}\left(\sin\theta\,\cancel{\sin\phi}\cos\phi\cos\theta - \sin\theta\cancel{\cos\phi}\right)$$

$$-d_{22}\left(\cos^2\phi\cos\theta - \sin^2\phi\cos\theta\right)\cos\phi\cos\theta$$

$$-d_{22}\left(-\cos\phi\sin\phi\cos\theta - \sin\phi\cos\phi\cos\theta\right)\sin\phi\cos\theta$$

$$-d_{22}\left(\cos^3\phi - \sin^2\phi\cos\phi\right)\cos^2\theta$$

$$+d_{22}\left(2\cos\phi\sin^2\phi\right)\cos^2\theta$$

$$= -d_{22}\left(\cos^3\phi - 3\cos\phi\sin^2\phi\right)\cos^2\theta$$

$$= -d_{22}\left(4\cos^3\phi - 3\cos\phi\right)\cos^2\theta$$

$$= -d_{22}\cos 3\phi\cdot\cos^2\theta$$

6a

Start with eq 6.97 in handout materials

$$\begin{vmatrix} (n_x \frac{\omega}{c})^2 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & (n_y \frac{\omega}{c})^2 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & (n_z \frac{\omega}{c})^2 - k_x^2 - k_y^2 \end{vmatrix} = 0$$

Crossing with x-axis, $k_x \neq 0$, $k_y = 0$, $k_z = 0$

$$\Rightarrow \begin{vmatrix} (n_x \frac{\omega}{c})^2 & 0 & 0 \\ 0 & (n_y \frac{\omega}{c})^2 - k_x^2 & 0 \\ 0 & 0 & (n_z \frac{\omega}{c})^2 - k_x^2 \end{vmatrix} = 0$$

Solutions:

1: $\frac{k_x}{\omega/c} = n_y$

2: $\frac{k_x}{\omega/c} = n_z$

In the same way you find
the crossings with the y- and z-axis.

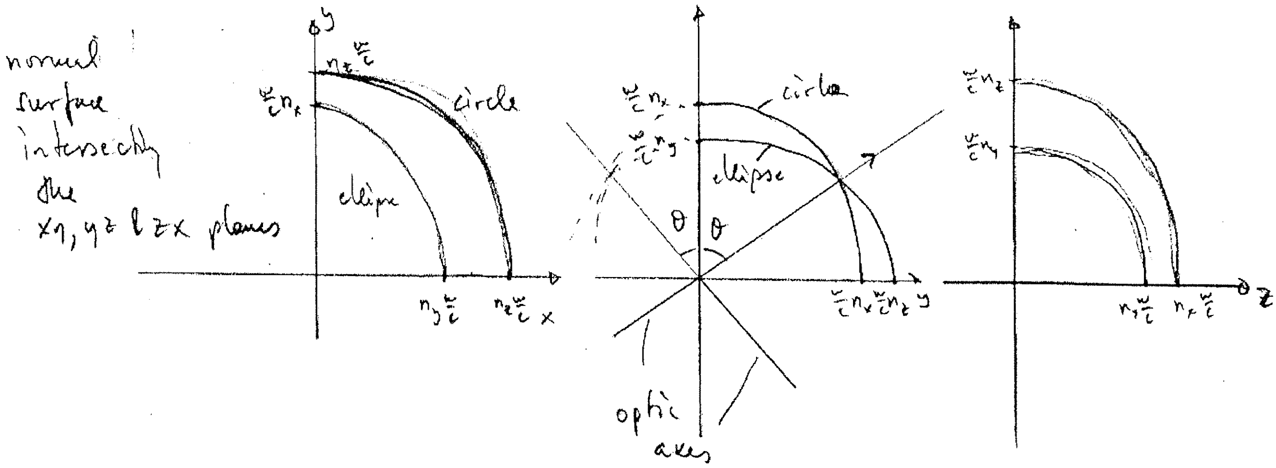
b)

By simply plotting the curves in the
x-y , x-z and y-z planes
you see immediately that the
curves will cross in the plane
containing the ~~##~~ highest and
lowest n. See also Mikael's
Compendium p. 89.

P3:

<u>Ans.</u> In the yz-plane ±54° from the z-axis.

Detailed solution



The normal surface(s) represent the solution(s) in terms of $\bar{k}$ or $\bar{s}$. It should be well known, easy to see that the optic axes are in the plane spanned by the extreme refractive indices (i.e. n_y & n_z → yz-plane). In this plane the ordinary ray (circle) is independent of θ $n = n_x$. The extraordinary ray follows the usual

$$\frac{1}{n_{eff}^2} = \frac{\cos^2\theta}{n_y^2} + \frac{\sin^2\theta}{n_z^2}$$ with θ defined as above.

Along optic axis extraordinary and ordinary ray propagates with the same speed. ⇒

$$\frac{1}{n_x^2} = \frac{\cos^2\theta}{n_y^2} + \frac{\sin^2\theta}{n_z^2}$$; a little algebra gives θ = 54°

7.

$$\frac{r_p}{r_s} = \frac{\epsilon\cos\phi - \sqrt{\epsilon - \sin^2\phi}}{\epsilon\cos\phi + \sqrt{\epsilon - \sin^2\phi}} \cdot \frac{\cos\phi + \sqrt{\epsilon - \sin^2\phi}}{\cos\phi - \sqrt{\epsilon - \sin^2\phi}}$$

$$= \frac{\epsilon\cos^2\phi - \cos\phi\sqrt{} + \epsilon\cos\phi\sqrt{} - \epsilon + \sin^2\phi}{\epsilon\cos^2\phi + \cos\phi\sqrt{} - \epsilon\cos\phi\sqrt{} - \epsilon + \sin^2\phi}$$

$$= \frac{\sqrt{}\cos\phi(\epsilon-1) - \sin^2\phi(\epsilon-1)}{\sqrt{}\cos\phi(1-\epsilon) + \sin^2\phi(1-\epsilon)}$$

$$= \frac{\sin\phi\,\mathrm{tg}\,\phi - \sqrt{\epsilon - \sin^2\phi}}{\sin\phi\,\mathrm{tg}\,\phi + \sqrt{\epsilon - \sin^2\phi}} = \rho$$

$$\Rightarrow \quad \frac{\sqrt{\epsilon - \sin^2\phi}}{\sin\phi\,\mathrm{tg}\,\phi} = \frac{1-\rho}{1+\rho}$$

$$\Rightarrow \quad \epsilon = \sin^2\phi + \sin^2\phi\,\mathrm{tg}^2\phi\left(\frac{1-\rho}{1+\rho}\right)^2$$

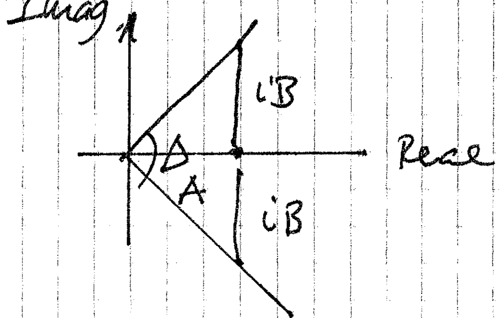
b)

For an inner reflection we replace

$N^2 = \epsilon$ by the inverse

$$\Rightarrow \frac{r_p}{r_s} = \frac{\sin\phi \, tg\phi - \sqrt{\frac{1}{N^2} - \sin^2\phi}}{\sin\phi \, tg\phi + \sqrt{\frac{1}{N^2} - \sin^2\phi}}$$

$$= \frac{N \sin\phi \, tg\phi - i\sqrt{N^2 \sin^2\phi - 1}}{N \sin\phi \, tg\phi + i\sqrt{N^2 \sin^2\phi - 1}} = \frac{A - iB}{A + iB}$$



We see directly that $tg\frac{\Delta}{2} = \frac{B}{A} = \frac{\sqrt{N^2 \sin^2\phi - 1}}{N \sin\phi \, tg\phi}$

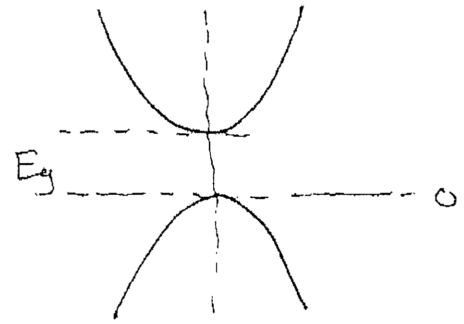
$\Delta = 45°$ for each reflection

Solve for N

$$N^2 = \frac{1}{\sin^2\phi \left(1 - tg^2\frac{\Delta}{2} \, tg^2\phi\right)} = 2.29$$

$$\underline{\underline{N = 1.514}}$$

Oppgave 3



Energibåndene er gitt av

$$E_v = -\frac{\hbar^2 k^2}{2m_h} \qquad E_c = E_g + \frac{\hbar^2 k^2}{2m_c}$$

$$E_c - E_v = E_g + \frac{\hbar^2 k^2}{2m_r} \qquad \frac{1}{m_r} = \frac{1}{m_e} + \frac{1}{m_v}$$

Fra den oppgitte ligningen

$$\epsilon_2 \sim \frac{JDS}{\omega^2} \sim \frac{1}{\omega^2} \int d^3k \, \delta\left(E_g + \frac{\hbar^2 k^2}{2m_r} - \hbar\omega\right)$$

$$= \frac{4\pi}{\omega^2} \int k^2 dk \, \delta\left(E_g + \frac{\hbar^2 k^2}{2m_r} - \hbar\omega\right)$$

Fra $\int g(x)\, \delta(f(x))\, dx = \frac{g(x_0)}{\frac{\partial f}{\partial x}\big|_{x_0}}$ fås

$$\epsilon_2 \sim \frac{1}{\omega^2} \frac{k_0^2}{k_0} \qquad \text{med } \hbar k_0 = \sqrt{2m_r(\hbar\omega - E_g)}$$

Dette gir til slutt:

$$\epsilon_2 \sim \frac{1}{\omega^2}\sqrt{\hbar\omega - E_g} \qquad \hbar\omega > E_g$$

$$\epsilon_2 = 0 \qquad \hbar\omega < E_g$$

b) Her får vi

$$\epsilon_2 \sim \frac{1}{\omega^2}\int d^3k\, \alpha\, k^2\, \delta\left(E_g + \frac{\hbar^2 k^2}{2m_r} - \hbar\omega\right)$$

$$= \frac{4\pi\alpha}{\omega^2}\, \frac{k_0^4}{k_0/m_r} = 4\pi\alpha\, m_r \cdot \frac{k_0^3}{\omega^2}$$

med igjen $\hbar k_0 = \sqrt{2m_r(\hbar\omega - E_g)}$
som gir

$$\epsilon_2 \sim \frac{1}{\omega^2}\left(\hbar\omega - E_g\right)^{3/2} \qquad \hbar\omega > E_g$$

$$\epsilon_2 = 0 \qquad \hbar\omega < E_g$$