Norges teknisk–naturvitenskapelige universitet NTNU

Institutt for fysikk
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Contact during the exam:
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Exam in TFY4205 Quantum Mechanics II
Thursday, December 12, 2013
15:00–19:00

Allowed help: Alternativ B
Approved pocket calculator.
K. Rottman: *Matematisk formelsamling* (All editions)
O.H. Jahren og K.J. Knutsen: *Formelsamling i matematikk*.

This problem set consists of 3 pages.

Problem 1

The hamiltonian of a harmonic oscillator in one dimension is

$$H_0 = \frac{1}{2m}\, p^2 + \frac{m\omega^2}{2}\, x^2\ . \tag{1}$$

a) Use the relations

$$a = i\,\sqrt{\frac{1}{2m\hbar\omega}}\; p + \sqrt{\frac{m\omega}{2\hbar}}\; x\ , \tag{2}$$

and

$$a^\dagger = -i\,\sqrt{\frac{1}{2m\hbar\omega}}\; p + \sqrt{\frac{m\omega}{2\hbar}}\; x\ , \tag{3}$$

together with the commutator $[x, p] = i\hbar$ to derive the commutator

$$[a, a^\dagger] = 1 \tag{4}$$

between the creation and annihilation operators $a^\dagger$ and a.

b) Derive the energy spectrum E_n for the harmonic oscillator by solving the Schrödinger equation $H_0|n\rangle = E_n|n\rangle$ using the creation and annihilation operators $a^\dagger$ and a.

c) Find the matrix elements $\langle m|a|n\rangle$ and $\langle m|a^\dagger|n\rangle$.

d) Find the matrix elements $\langle m|x|n\rangle$ and $\langle m|p|n\rangle$.

e) Find the matrix elements $\langle m|x^2|n\rangle$ and $\langle m|p^2|n\rangle$.

f) Determine $\Delta x \Delta p$ for the harmonic oscillator in energy eigenstate state $|n\rangle$. We define

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 \,, \tag{5}$$

and correspondingly for Δp.

g) Consider the result for the diagonal elements $\langle n|x^2|n\rangle$. What does it have to say about the partitioning of the total energy E_n between the potential and kinetic energy?

Problem 2

We now add a perturbation to the harmonic oscillator hamiltonian. In the following, we ask for *exact* expressions, and not approximations.

a) If the perturbation is $H' = \alpha_1 x$, what is the energy spectrum of the perturbed hamiltonian $H_a = H_0 + H'$? Denote the energy eigenstates of H_a by $|n'\rangle$.

b) If the perturbation is $H" = \alpha_2 x^2$, what is the energy spectrum of the perturbed hamiltonian $H_b = H_0 + H"$? Denote the energy eigenstates of H_b by $|n"\rangle$.

c) How can one express $|n'\rangle$ and $|n"\rangle$ in terms of the eigenstates of H_0, $|n\rangle$?

In the coordinate representation we have that the wave function is given by

$$\psi_n(x) = \langle x|n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-m\omega x^2/2\hbar} \,, \tag{6}$$

where H_n is a hermite polynomial.

d) What is $\langle x|n'\rangle$?

e) What is $\langle x|n"\rangle$?

We now consider the Stark effect. An charge q is bound in a harmonic potential so that $H = H_0$. An electric field $\mathcal{E}$ is turned on. This adds the perturbation $H' = -qx\mathcal{E}$ to the hamiltonian.

f) Find the average dipole moment $\langle qx\rangle$ as a function of $\mathcal{E}$ when the perturbed harmonic oscillator is in an energy eigenstate $|n'\rangle$.

Problem 3

We consider perturbation theory in the following. Let $E_n^{(0)}$ be the energy of the unperturbed hamiltonian H_0. We add a perturbation λH_1 where λ is a small dimensionless number. The change in energy is then

$$E_n = E_n^{(0)} + \lambda \langle n^{(0)}|H_1|n^{(0)}\rangle + \lambda^2 \sum_{m\neq n} \frac{|\langle m^{(0)}|H_1|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}} + \mathcal{O}(\lambda^3) \,, \tag{7}$$

where $|n^{(0)}\rangle$ is the energy eigenstate corresponding to $E_n^{(0)}$ of the unperturbed hamiltonian H_0.

a) Calculate to second order in λ the change of the harmonic oscillator energy levels E_n when $H_1 = \alpha_1 x$?

b) Calculate to second order in λ the change of the harmonic oscillator energy levels E_n when $H_1 = \alpha_2 x^2$?

Problem 4

The lowest-order Born approximation to the scattering amplitude from a potential $V(\vec{r})$ is given by

$$f^B = -\frac{m}{2\pi\hbar^2} \int d^3\vec{r}\, V(\vec{r}) e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} \,. \tag{8}$$

Here $\vec{k}$ and $\vec{k}'$ point along the momentum direction of the scattered particle before and after the scattering event. We place the z-direction in such a way that it points along $\vec{k}$. In polar coordinates we then have $k'_x = k\sin\theta\cos\phi$, $k'_y = k\sin\theta\sin\phi$ and $k'_z = k\cos\theta$, where $k = |\vec{k}|$.

a) Find the scattering cross section $(d\sigma/d\Omega)$ in polar coordinates when

$$V(\vec{r}) = \frac{\alpha}{abc}\, e^{-(x/a)^2-(y/b)^2-(z/c)^2} \,, \tag{9}$$

where α, a, b and c are all positive constants, $a \geq b \geq c$ and $\vec{r} = (x, y, z)$.

(Hint: $\int_{-\infty}^{+\infty} \exp(-x^2)dx = \sqrt{\pi}$.)

b) What do the equipotential surfaces of V look like when

1) $a > b > c$?
2) $a = b > c$?
3) $a = b = c$?

c) Difficult last question: Interpret $d\sigma/d\Omega$ in all three cases.