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Exam in TFY4205 Quantum Mechanics II
Saturday, December 20, 2014
09:00–13:00

Allowed help: Alternativ C

This problem set consists of 4 pages.

Problem 1

We will in this problem consider an electron with charge e and mass m constrained to move in the (x, y) plane limited by $0 \leq x \leq L$ and $0 \leq y \leq W$. We assume that the system is periodic in the x direction.
There is a constant magnetic field perpendicular to the the plane, $\vec{B} = B\vec{e}_z$. Associated with the magnetic field, there is a vector potential $\vec{A}$, and we have that $\vec{B} = \vec{\nabla} \times \vec{A}$. The hamiltonian (1) then becomes

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 \ . \tag{1}$$

The standard commutation relations apply,

$$[x, p_x] = [y, p_y] = i\hbar \ , \tag{2}$$

and

$$[x, y] = [x, p_y] = [y, p_x] = [p_x, p_y] = 0 \ . \tag{3}$$

In the following, we choose the Landau gauge, $\vec{A} = (-By, 0, 0)$.
Let us now define two sets of variables,

$$\xi = \frac{1}{eB}\left(p_y + eA_y\right) \ , \tag{4}$$

$$\eta = -\frac{1}{eB}\left(p_x + eA_x\right) \ , \tag{5}$$

and

$$X = x - \xi \ , \tag{6}$$

$$Y = y - \eta \ . \tag{7}$$

ξ and η are called the *relative coordinates* and X and Y are called the *guiding center* coordinates.

a) Show that

$$[X, Y] = -[\xi, \eta] = i\left(\frac{\hbar}{eB}\right) = il^2 \ , \tag{8}$$

where we have defined the magnetic length

$$l = \sqrt{\frac{\hbar}{eB}} \ . \tag{9}$$

Furthermore, show that

$$[\xi, X] = [\eta, Y] = [\xi, Y] = [\eta, X] = 0 \ . \tag{10}$$

Comment on the commutation relations in (8): what do they say about the relation between ξ and η, and X and Y?

b) Show that the hamiltonian, (1) can be written

$$H = \frac{m}{2}\,\omega^2\left(\xi^2 + \eta^2\right) \ , \tag{11}$$

where we have defined the cyclotron frequency

$$\omega = \frac{eB}{m} \ . \tag{12}$$

Show that this is the *harmonic oscillator* hamiltonian in disguise. Try to give a physical interpretation of the motion of the harmonic oscillator relative to the guiding center coordinates, (X, Y).

This implies that the energy levels of the system is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \ , \tag{13}$$

where $n = 0, 1, 2, \cdots$. These are the *Landau levels* (which we now have derived by a different route than in the text book — this one being due to Ryogo Kubo).

c) We now construct the energy eigenfunctions for this system. Show that

$$[X, H] = [Y, H] = 0 \ , \tag{14}$$

where H is given in equation (1). This implies that the energy eigenfunctions may also be eigenfunctions of either X or Y, but not both. Why is this?

We have split the coordinates x and y into two pairs, (ξ, η) and (X, Y). We will express the wave function in terms of the variables X and η. Why can we not use all four at the same time?

We choose our energy eigenfunctions also to be eigenfunctions of Y. Hence, we will construct $\psi_{\Upsilon,n}(X, \eta)$ such that

$$H\psi_{\Upsilon,n}(X, \eta) = \hbar\omega\left(n + \frac{1}{2}\right)\psi_{\Upsilon,n}(X, \eta) \ , \tag{15}$$

and

$$Y\psi_{\Upsilon,n}(X,\eta) = \Upsilon\psi_{\Upsilon,n}(X,\eta)\ . \tag{16}$$

Show that the eigenfunctions we seek are

$$\psi_{\Upsilon,n}(X,\eta) = \frac{e^{i\Upsilon X/l^2}}{\sqrt{L}}\phi_n(\eta)\ , \tag{17}$$

where ϕ_n is the harmonic oscillator energy eigenfunction.

We stated that the system is periodic in the x direction with periodicity L. It must then also be periodic in X.Hence, we must have that

$$\psi_{\Upsilon,n}(X+L,\eta) = \psi_{\Upsilon,n}(X,\eta)\ . \tag{18}$$

Show that this periodicity leads to

$$\Upsilon = \frac{2\pi l^2}{L}\ k\ , \tag{19}$$

where k is an integer.

The system has a width (in the y direction) W. Show that this leads to the inequalities

$$0 < k < \frac{WL}{2\pi l^2}\ . \tag{20}$$

Each energy level n is then degenerate. Show that the degeneration is given by

$$\frac{\Phi_{tot}}{h/e}\ , \tag{21}$$

where $\Phi_{tot} = WLB$ is the total magnetic flux through the system. We ignore here the degeneration due to electron spin.

d) So far we have considered one single electron. Now we assume that there are many in the system. We ignore interactions between them.

Electrons are fermions and obey the Pauli principle: only one electron in each state. As the system is degenerate with respect to the electron spin, we may place two electrons in each Landau level n. Suppose we have a two-dimensional electron density ρ, what magnetic field $B = B_0$ will ensure that all electrons will be in the first Landau level? Plot the *Fermi energy* — the energy of the most energetic electron — as a function of magnetic field B. Explain why the figure looks like what it does.

Problem 2

The lowest-order Born approximation to the scattering amplitude from a potential $V(\vec{r})$ is given by

$$f^B = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\ V(\vec{r})e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}}\ . \tag{22}$$

Here $\vec{k}$ and $\vec{k}'$ point along the momentum direction of the scattered particle before and after the scattering event. We place the z-direction in such a way that it points along $\vec{k}$. In polar coordinates we then have $k'_x = k\sin\theta\cos\phi$, $k'_y = k\sin\theta\sin\phi$ and $k'_z = k\cos\theta$, where $k = |\vec{k}|$.

a) We assume a potential

$$V(\vec{r}) = \frac{\alpha}{\pi bc}\, e^{-(y/b)^2-(z/c)^2}\delta(x)\ , \tag{23}$$

where α and $b \geq c$ are positive constants and $\delta(x)$ is the Dirac delta-function. Furthermore, $\vec{r} = (x, y, z)$. Sketch the equipotential surfaces of V when

1) $b > c$?
2) $b = c$?

b) Show that

$$\int d^3\vec{r}V(\vec{r})e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} = \alpha e^{-(b/2)^2(k_y-k_y')^2-(c/2)^2(k_z-k_z)^2}\ . \tag{24}$$

(Hint: $\int_{-\infty}^{+\infty} \exp(-x^2)dx = \sqrt{\pi}$ — and *think Cartesian!*)

c) Find the scattering cross section $(d\sigma/d\Omega)$ expressed in polar coordinates.