Norges teknisk–naturvitenskapelige universitet NTNU

Institutt for fysikk
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Exam in TFY4205 Quantum Mechanics II

Saturday, August 8, 2015
09:00–13:00

Allowed help: Alternativ C

This problem set consists of 2 pages, plus an Appendix of one page.

Problem 1

We will in this problem consider a pendulum of mass m at the end of a massless rod of length l moving about a pivot P. There is a gravitational field g pointing downwards in the vertical direction. The pendulum moves in a fixed plane normal to the vertical direction and the angle the rod makes with the vertical in this plane is θ.

a) Find the energy levels of the pendulum in the small-θ approximation.

b) Find the lowest order correction to the ground state resulting from the inaccuracy of the small-angle approximation.

Problem 2

The Hamiltonian for a spinnless charged particle in a magnetic field is

$$H = \frac{1}{2m}\left[\vec{p} - q\vec{A}\right]^2 \ , \tag{1}$$

where m is the mass, q the charge, $\vec{p}$ is the momentum operator and $\vec{A}$ is related to the magnetic field by

$$\vec{B} = \vec{\nabla} \times \vec{A} \ . \tag{2}$$

a) Show that the gauge transformation

$$\vec{A}(\vec{r}) \to \vec{A}(\vec{r}) + \vec{\nabla} f(\vec{r}) \tag{3}$$

is equivalent to multiplying the wave function to a factor $\exp(iqf(\vec{r})/\hbar)$. What is the significance of this result?

b) Consider the case of a uniform magnetic field $\vec{B}$ directed along the z-axis. Show that the energy levels can be written as

$$E = \left(n + \frac{1}{2}\right)\frac{|q|\hbar}{m}B + \frac{\hbar^2 k_z^2}{2m} \ , \tag{4}$$

where $n = 0, 1, 2, \ldots$ is a discrete quantum number and $\hbar k_z$ is the (continuous) momentum is the z-direction.

Discuss the nature of the wave functions.

Hint: Use the gauge where $A_x = -By$, $A_y = A_z = 0$.

Problem 3

We will in this problem consider scattering.

a) Give an interpretation of the differential cross section $d\sigma/d\Omega$.

In the rest of this problem, we will consider scattering as a stationary problem. We will consider scattering of particles with mass m on a potential $V(\vec{r})$. The incoming wave function is $\psi_{in}(\vec{r}) = \exp(i\vec{k} \cdot \vec{r})$. We assume that $V(\vec{r})$ falls off fast enough with increasing $r = |\vec{r}|$ so that the scattered wave function may be written

$$\psi_{sc}(\vec{r}) = \psi(\vec{r}) - \exp(i\vec{k} \cdot \vec{r}) \approx f(\theta, \phi) \frac{\exp(ikr)}{r} \ , \tag{5}$$

where

$$f(\theta, \phi) = -\frac{1}{4\pi} \int d^3r' \ \exp(-i\vec{k}_f \cdot \vec{r}')U(\vec{r}')\psi(\vec{r}') \tag{6}$$

is the scattering amplitude, $\psi(\vec{r})$ is the total wave function, $k = |\vec{k}|$, $\vec{k}_f = k \ \vec{r}/r$ and $U(\vec{r}) = 2mV(\vec{r})/\hbar^2$.

b) What is the scattering amplitude $f^B(\theta, \phi)$ in the first Born approximation for a general potential $V(\vec{r})$? Show that for a centrally symmetric potential, $V(\vec{r}) = V(r)$, the first Born approximation may be expressed as

$$f^B(\theta) = f^B(q) = -\frac{2m}{\hbar^2 q} \int_0^\infty dr \ r \ \sin(qr)V(r) \ , \tag{7}$$

where $q = |\vec{q}|$ and $\vec{q} = \vec{k}_f - \vec{k}$.

c) Calculate $f^B(q)$ and $d\sigma^B/d\Omega$ for particles with mass m and energy $E = \hbar^2 k^2/2m$ that are scattered on the potential

$$V(r) = \begin{cases} V_0 & \text{for } r \leq a \ , \\ 0 & \text{for } r > a \ . \end{cases} \tag{8}$$

Assume here that

$$\frac{d\sigma^B}{d\Omega} = |f^B(q)|^2 \ . \tag{9}$$

The following information may be of some use:

The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \ , \tag{10}$$

where x is the position, m is the mass and ω is the oscillator frequency. The energy levels are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \ . \tag{11}$$

The ground state wave function for a harmonic oscillator is

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}\ x^2\right) \ . \tag{12}$$

Here are some useful integrals,

$$\int_0^\infty d\phi\ \exp\left(-\lambda\phi^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \ , \tag{13}$$

$$\int \phi \sin(a\phi)\ d\phi = \frac{1}{a^2}\sin(a\phi) - \frac{\phi}{a}\cos(a\phi) + C \ , \tag{14}$$

and

$$\int_0^\infty \phi \sin(b\phi)\exp(-c\phi^2)d\phi = \frac{\sqrt{\pi}}{4} b c^{-3/2}\exp(-b^2/4c) > 0 \ . \tag{15}$$

The following series may also be useful,

$$\sin(\phi) = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \cdots \ , \tag{16}$$

$$\cos(\phi) = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots \ , \tag{17}$$

and

$$\exp(\phi) = 1 + \phi + \frac{\phi^2}{2!} + \cdots \ . \tag{18}$$