ENGLISH

Information about the exam
Read each problem carefully. The maximum number of points you can get on this exam is 100. The maximum number of points each problem can give (perfect score) is indicated in paranthesis. Good luck.

Problem 1 (5 points)
Let $|n\rangle$ be energy eigenstates for a quantum mechanical harmonic oscillator and let $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ be a coherent state. Compute what c_n has to be in order for both $a|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\alpha\rangle = 1$ to be fulfilled. *Hint:* you may find some of these properties useful: $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $[a,a^\dagger] = 1$.

Problem 2 (12 points)
The stationary Schrödinger equation reads:

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle \tag{1}$$

where $\hat{H} = \hat{H}_0 + \lambda\hat{H}_1$. Assume that $\lambda\hat{H}_1$ is a perturbative term and that the eigenvalues and eigenfunctions of $\hat{H}_0$ are known. The eigenvalues and eigenstates can be expanded as follows:

$$\begin{aligned} E_n &= E_n^{(0)} + \lambda E_n^{(1)} + \dots \\ |\psi_n\rangle &= |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \dots, \end{aligned} \tag{2}$$

where $E_n^{(0)}$ and $|\psi_n^{(0)}\rangle$ are defined by the unperturbed problem:

$$H_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle. \tag{3}$$

Assume that the unperturbed energy level $E_n^{(0)}$ is non-degenerate. Using this information, find an expression for the lowest-order correction to the state of the system, $\lambda|\psi_n^{(1)}\rangle$, expressed via the unperturbed eigenstates, the unperturbed eigenvalues, and the perturbation in the Hamilton-operator. *Hint:* if we demand that the total state $|\psi_n\rangle$ should be normalized to 1 up to order $O(\lambda)$ as well, this will tell you something about the value of $\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle$.

Problem 3 (18 points)
For each of the approximation methods below, explain under which circumstances they can be expected to give useful results. In particular, explain the mathematical reason for why they are expected to be useful only in certain cases. In effect, how are we mathematically approximating the system in order to use the methods? You should use both words and equations in your answer, as you are expected to outline the key mathematical ideas of these methods (without giving any full derivations as we have done in the lectures).

- WKB-approximation.
- Adiabatic approximation.
- Variational method.

Problem 4 (12 points)
(a) Consider a 2D plane (xy) with area $A = L_xL_y$ where free electrons move around. Apply a magnetic field $\mathbf{B}$ in the z-direction and write down the resulting Hamilton-operator describing an electron in this plane. You can ignore the electron spin.

(b) Derive an analytical expression for the energy eigenvalues of $\hat{H}$.

Problem 5 (4 points)
Consider a region of space where $\mathbf{B} = 0$. Does this mean that we are free to set the magnetic vector potential $\mathbf{A} = 0$ everywhere? Explain your answer in detail.

Problem 6 (3 points)
State the adiabatic theorem in quantum mechanics as accurately as possible.

Problem 7 (3 points)
The total scattering cross section σ computed quantum mechanically for low-energy scattering on a hard sphere with radius R is $\sigma = 4\pi R^2$. Classically, the result is $\sigma = \pi R^2$. Give a qualitative physical argument for why there is an extra factor of 4 in the QM case.

Problem 8 (5 points)
The optical theorem $\sigma = \frac{4\pi}{k}\text{Im}\{f(\theta = 0)\}$ relates the total scattering cross section to the imaginary part of the forward-scattering amplitude $f(\theta = 0)$ where f is defined via the asymptotic form of the wavefunction in the scattering problem:

$$\psi \simeq e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)e^{ikr}/r. \tag{4}$$

Explain physically why it is reasonable that it is $\theta = 0$ that contributes in the optical theorem and why we need to take the imaginary part.

Problem 9 (5 points)
Under which circumstances is the method of partial waves for scattering expected to be useful? Explain the essential idea behind this method using both words and, to the extent you are able, equations. You are not expected to recall the exact form of all relevant equations, but you are expected to be able to roughly sketch the key idea behind this method.

Problem 10 (5 points)
Under which circumstances is the Born-approximation for scattering expected to be good? Why is the result for the scattering amplitude f in the Born-approximation not consistent with the optical theorem?

Problem 11 (12 points)
The quantum mechanical expression for the operator associated with the magnetic vector potential is:

$$\hat{\mathbf{A}} = \sum_{\mathbf{k},\lambda} \mathbf{e}_{\mathbf{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}(a_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t} + a^\dagger_{\mathbf{k},\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}). \tag{5}$$

Using that $\langle\alpha|a_{\mathbf{k},\lambda}|\alpha\rangle = \alpha$ and $\langle\alpha|a^\dagger_{\mathbf{k},\lambda}|\alpha\rangle = \alpha^*$, compute the standard deviation $\Delta E = \sqrt{\langle\hat{\mathbf{E}}^2\rangle - \langle\hat{\mathbf{E}}\rangle^2}$ for the electric field operator $\hat{\mathbf{E}}$ in a coherent state $|\alpha\rangle$ for mode $(\mathbf{k},\lambda)$. *Hint:* we defined a coherent state and the essential properties of the operators a and $a^\dagger$ in Problem 1.

Problem 12 (3 points)
What is the difference between stimulated and spontaneous emission in fully quantized radiation theory? Mention a technological device for which stimulated emission is of crucial importance.

Problem 13 (3 points)
Explain clearly the difference between a mixed state and a superposition of states. Under which circumstances is $\text{Tr}(\rho) = 1$ where ρ is the density matrix operator of a system?

Problem 14 (10 points)
What is the resistivity of a material at temperature $T = 0$ where electrons move through an ideal periodic crystal-structure? Explain what the Kronig-Penney model is and how it gives rise to gaps in the allowed energy eigenvalues. Under what circumstances does the Kronig-Penney model describe a metal, a semiconductor, or an insulator?

You may certainly use figures and, if possible, equations. You are not expected to be able to give the full derivation of the energy eigenvalues or remember the exact form of the Kronig-Penney potential. You are, however, expected to be able to outline some of the key results and explain clearly how one can, in principle, obtain these results.