Final exam in TFY 4205 Quantum Mechanics II Fall 2013

Solutions:

Problem 1

a) $$\underline{\underline{[a,a^+]}} = \left[i\sqrt{\frac{1}{2m\hbar\omega}}\,p + \sqrt{\frac{m\omega}{2\hbar}}\,x, \; -i\sqrt{\frac{1}{2m\hbar\omega}}\,p + \sqrt{\frac{m\omega}{2\hbar}}\,x \right]$$

$$= i\frac{1}{2\hbar}[p,x] - i\frac{1}{2\hbar}[x,p]$$

$$= -i\frac{1}{2\hbar}i\hbar - i\frac{1}{2\hbar}i\hbar \underline{\underline{= 1}}$$

b) $$\left.\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\,(a^+ + a) \\ p &= i\sqrt{\frac{\hbar\omega m}{2}}\,(a^+ - a) \end{aligned}\right\}$$

$$\underline{H_0} = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} x^2$$

$$= -\frac{1}{2m} \frac{\hbar m\omega}{2} (a^+ - a)(a^+ - a)$$

$$+ \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} (a^+ + a)(a^+ + a)$$

$$= \frac{\hbar\omega}{2} (a^+ a + a a^+) = \underline{\hbar\omega (a^+ a + \tfrac{1}{2})}$$

Let $a^+ a |n\rangle = n |n\rangle$.

Then $H_0 |n\rangle = \underbrace{\hbar\omega (n + \tfrac{1}{2})}_{E_n} |n\rangle$

We must find $\underline{n}$.

We have $\underline{[a^+ a, a]} = \underbrace{a^+ a}_{a a^+ - 1} a - a a^+ a = \underline{-a}$

Hence,

$$a^+ a (a|n\rangle) = a(a^+ a |n\rangle) - a|n\rangle$$

$$= (n-1) a |n\rangle$$

In general:

$$a^+ a \left(a^k |n\rangle\right) = (n-k)\, a^k |n\rangle$$

This is an integer.

$\Rightarrow$
$a^k|n\rangle$ is an eigenvector of a^+a with eigenvalue $(n-k)$.

The norm is always positive:

$$\left(\langle n| a^+\right)\left(a|n\rangle\right) = n\langle n|n\rangle \geq 0 \Rightarrow$$

$\underline{(n-k) \geq 0}$ $\Rightarrow$ 1. $\underline{n \text{ is an integer.}}$

2. $\underline{a|n=0\rangle = 0\,|n=0\rangle}$

Hence,

$$\underline{\underline{H_0|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle}}$$

$\underline{\underline{\text{where } \quad n = 0, 1, 2, 3, \dots}}$

c) In the following, we assume $t=0$.

We assume $a|n\rangle = C|n-1\rangle$

where $\langle n|n\rangle = \langle n-1|n-1\rangle = 1$.

$$\left.\begin{aligned}\langle n|a^\dagger a|n\rangle &= n\langle n|n\rangle = n\\ \langle n|a^\dagger a|n\rangle &= \left(\langle n|a^\dagger\right)\left(a|n\rangle\right) = C^2\langle n-1|n-1\rangle\\ &= C^2\end{aligned}\right\}$$

$$\Rightarrow \underline{C = \sqrt{n}}$$

$\langle m|n\rangle = \delta_{m,n}$

$\Rightarrow$

$$\underline{\underline{\langle m|a|n\rangle}} = \sqrt{n}\,\langle m|n-1\rangle = \underline{\underline{\sqrt{n}\,\delta_{m,n-1}}}$$

$$\underline{\underline{\langle m|a^\dagger|n\rangle}} = \langle n|a|m\rangle^\dagger = \sqrt{m}\,\delta_{n,m-1}$$

$$= \underline{\underline{\sqrt{n+1}\,\delta_{m,n+1}}}$$

d) $\underline{\underline{\langle m|x|n\rangle}} = \sqrt{\frac{\hbar}{2m\omega}}\left\{\langle m|a^{+}|n\rangle + \langle m|a|n\rangle\right\}$

$$= \underline{\underline{\sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right\}}}$$

$\underline{\underline{\langle m|p|n\rangle}} = i\sqrt{\frac{\hbar\omega m}{2}}\left\{\langle m|a^{+}|n\rangle - \langle m|a|n\rangle\right\}$

$$= \underline{\underline{i\sqrt{\frac{\hbar\omega m}{2}}\left\{\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right\}}}$$

e) $\underline{\underline{\langle m|x^2|n\rangle}} = \sum_{k=0}^{\infty}{}' \langle m|x|k\rangle\langle k|x|n\rangle$

$$= \sum_{k=0}^{\infty}{}' \frac{\hbar}{2m\omega}\left\{\sqrt{k+1}\,\delta_{m,k+1} + \sqrt{k}\,\delta_{m,k-1}\right\}$$

$$\left\{\sqrt{n+1}\,\delta_{k,n+1} + \sqrt{n}\,\delta_{k,n-1}\right\}$$

$$= \underline{\underline{\frac{\hbar}{2m\omega}\left\{\sqrt{(n+2)(n+1)}\,\delta_{m,n+2}\right.}}$$

$$\underline{\underline{\left. + (2n+1)\,\delta_{m,n} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right\}}}$$

$$\underline{\underline{\langle m|p^2|m\rangle}} = \sum_{k=0}^{\infty}{}' \langle m|p|k\rangle\langle k|p|m\rangle$$

$$= -\frac{\hbar m\omega}{2}\sum_{k=0}^{\infty}{}' \left\{\sqrt{k+1}\,\delta_{m,k+1} - \sqrt{k}\,\delta_{m,k-1}\right\}$$

$$\left\{\sqrt{m+1}\,\delta_{k,m+1} - \sqrt{m}\,\delta_{k,m-1}\right\}$$

$$\underline{\underline{= -\frac{\hbar m\omega}{2}\left\{\sqrt{(m+2)(m+1)}\,\delta_{m,m+2}\right.}}$$

$$\underline{\underline{\left. -(2m+1)\,\delta_{m,k} + \sqrt{m(m-1)}\,\delta_{m,m-2}\right\}}}$$

f)

$$\left.\begin{aligned}\langle m|x|m\rangle &= 0\\ \langle m|p|m\rangle &= 0\end{aligned}\right\}\quad \underline{\langle x\rangle^2 = \langle p\rangle^2 = 0}$$

$$\langle m|x^2|m\rangle = \frac{\hbar}{m\omega}\left(m+\tfrac{1}{2}\right) \Rightarrow \underline{\Delta x = \sqrt{\frac{\hbar}{m\omega}\left(m+\tfrac{1}{2}\right)}}$$

$$\langle m|p^2|m\rangle = \hbar m\omega\left(m+\tfrac{1}{2}\right) \Rightarrow \underline{\Delta p = \sqrt{\hbar m\omega\left(m+\tfrac{1}{2}\right)}}$$

$$\underline{\underline{\Delta x\,\Delta p = \hbar\left(m+\tfrac{1}{2}\right)}}$$

g) $\langle H_0 \rangle = \hbar\omega(n+\frac{1}{2})$

$$\langle H_0 \rangle = \frac{1}{2m}\langle p^2 \rangle + \frac{m\omega^2}{2}\langle x^2 \rangle$$

$$\left.\begin{aligned} \frac{1}{2m}\langle p^2 \rangle &= \frac{1}{2}\hbar\omega(n+\tfrac{1}{2}) \\ \frac{m\omega^2}{2}\langle x^2 \rangle &= \frac{1}{2}\hbar\omega(n+\tfrac{1}{2}) \end{aligned}\right\}$$

We see that the expectation value of the kinetic and potential energies each are <u>one half the expectation value of the total energy</u>.

<u>Problem 2</u>

a) $H_a = H_0 + \alpha_1 x = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \alpha_1 x$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x + \frac{\alpha_1}{m\omega^2}\right)^2 - \frac{\alpha_1^2}{2m\omega^2}$$

$$\Rightarrow \quad H_a |n'\rangle = E^a_{n'} |n'\rangle$$

$$\underline{\underline{E^a_{n'} = \hbar\omega\left(n' + \frac{1}{2}\right) - \frac{\alpha_1^2}{2m\omega^2}}}$$

$$\underline{\underline{n' = 0, 1, 2, \ldots}}$$

b) $$H_b = H_0 + \alpha_2 x^2 = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \alpha_2 x^2$$

$$= \frac{p^2}{2m} + \frac{m}{2}\left(\omega^2 + \frac{2\alpha_2}{m}\right) x^2$$

$$\Rightarrow \quad H_b |n''\rangle = E^b_{n''} |n''\rangle$$

$$\underline{\underline{E^b_{n''} = \hbar\sqrt{\omega^2 + \frac{2\alpha_2}{m}}\left(n'' + \frac{1}{2}\right)}}$$

$$\underline{\underline{n'' = 0, 1, 2, \ldots}}$$

c) $$\underline{\underline{|n'\rangle}} = \sum_{n=0}^{\infty} |n\rangle\langle n|n'\rangle = \underline{\underline{\sum_{n=0}^{\infty} d_{nn'} |n\rangle}}$$

$$\underline{\underline{|n''\rangle}} = \sum_{n=0}^{\infty} |n\rangle\langle n|n''\rangle = \underline{\underline{\sum_{n=0}^{\infty} \delta_{nn''} |n\rangle}}$$

d)

$$\langle x|n'\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^{n'} n'!}} H_{n'}\left(\sqrt{\frac{m\omega}{\hbar}}\left(x+\frac{\alpha_1}{m\omega^2}\right)\right) e^{-m\omega\left(x+\frac{\alpha_1}{m\omega^2}\right)^2/2\hbar}$$

e)

$$\langle x|n''\rangle = \left(\frac{m\omega''}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^{n''} n''!}} H_{n''}\left(\sqrt{\frac{m\omega''}{\hbar}}\, x\right) e^{-m\omega'' x^2/2\hbar}$$

where $\omega'' = \sqrt{\omega^2 + \frac{2\alpha_2}{m}}$

f) We have from 1 d that $\langle n|x|n\rangle = 0$.

Hence, we now have

$$\langle n'|\left(x+\frac{\alpha_1}{m\omega^2}\right)|n'\rangle = 0 \quad \Rightarrow$$

$$\langle n'|x|n'\rangle = -\frac{\alpha_1}{m\omega^2} = \frac{q\,\mathcal{E}}{m\omega^2}$$

where $\alpha_1 = -q\,\mathcal{E}$

The average dipole moment is therefore

$$\underline{\underline{\langle n' | q x | n' \rangle = \frac{q^2}{m\omega^2} \mathcal{E}}}$$

Problem 3

a) We set

$$|n\rangle = |n^{(0)}\rangle$$

$$\langle n | H_1 | n \rangle = \alpha_1 \langle n | x | n \rangle = 0$$

$$\sum_{m \neq n}' \frac{|\langle m | H_1 | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{\alpha_1^2}{2m\omega^2} \left\{ \frac{(\sqrt{n+1})^2}{-1} + \frac{(\sqrt{n})^2}{1} \right\}$$

$$= \frac{\alpha_1^2}{2m\omega^2} \left(n - n - 1 \right) = -\frac{\alpha_1^2}{2m\omega^2}$$

$$\Rightarrow \quad \underline{\underline{E_n = \hbar\omega(n + \tfrac{1}{2}) - \frac{\alpha_1^2}{2m\omega^2} \lambda^2 + O(\lambda^3)}}$$

Compare this with 2a.

b)

$$\underline{\langle n | H_1 | n \rangle} = \alpha_2 \frac{\hbar}{2m\omega} (2n+1)$$

$$= \underline{\alpha_2 \frac{\hbar}{m\omega} \left(n+\frac{1}{2}\right)}$$

(From 1c).

$$\sum_{m \neq n}' \frac{|\langle m | H_1 | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \alpha_2^2 \left(\frac{\hbar}{2m\omega}\right)^2 \left\{ \frac{(n+2)(n+1)}{\hbar\omega\,(-2)} + \frac{n(n-1)}{\hbar\omega\, 2} \right\} = -\alpha_2^2 \frac{\hbar\omega}{2m^2\omega^4} \left(n+\frac{1}{2}\right)$$

Hence, we have

$$\underline{E_n} = \hbar\omega\left(n+\frac{1}{2}\right) + \lambda\alpha_2 \frac{\hbar\omega}{m\omega^2}\left(n+\frac{1}{2}\right) - \lambda^2\alpha_2^2 \frac{\hbar\omega}{2(m\omega^2)^2}\left(n+\frac{1}{2}\right) + O(\lambda^3)$$

$$= \underline{\underline{\hbar\omega\left(n+\frac{1}{2}\right)\left(1 + \frac{\lambda\alpha_2}{m\omega^2} - \frac{1}{2}\left(\frac{\lambda\alpha_2}{m\omega^2}\right)^2 + O(\lambda^3)\right)}}$$

Compare this expression to the exact one in 2b: Expand $\sqrt{\omega + \frac{2\alpha_2}{m}}$ in power of α_2!

Problem 4

a) Use Cartesian coordinates:

$$\int_{-\infty}^{+\infty} dx\, e^{-\left(\frac{x}{a}\right)^2 - i(k_x - k_x')x} = a\sqrt{\pi}\, e^{-\frac{a^2}{4}(k_x - k_x')^2}$$

Hence,

$$\int d^3r\, V(\vec{r})\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} = \pi^{3/2}\alpha\, e^{-\frac{a^2}{4}(k_x-k_x')^2 - \frac{b^2}{4}(k_y-k_y')^2 - \frac{c^2}{4}(k_z-k_z')^2}$$

$$= \pi^{3/2}\alpha\, e^{-\frac{a^2}{4}q_x^2 - \frac{b^2}{4}q_y^2 - \frac{c^2}{4}q_z^2}$$

Where we have introduced $\vec{q} = \vec{k} - \vec{k}'$.

$$\left.\begin{aligned} q_x &= k \sin\theta\cos\phi \\ q_y &= k \sin\theta\sin\phi \\ q_z &= k(1-\cos\theta) \end{aligned}\right\}$$

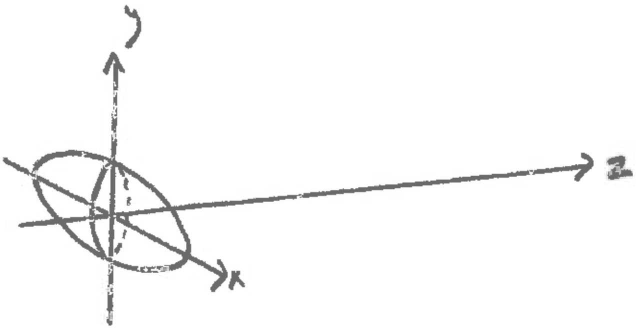
$$f^B = -\frac{m\alpha\sqrt{\pi}}{2\hbar^2}\, e^{-\frac{k^2}{4}\left(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2(1-\cos\theta)^2\right)}$$

The differential cross section is

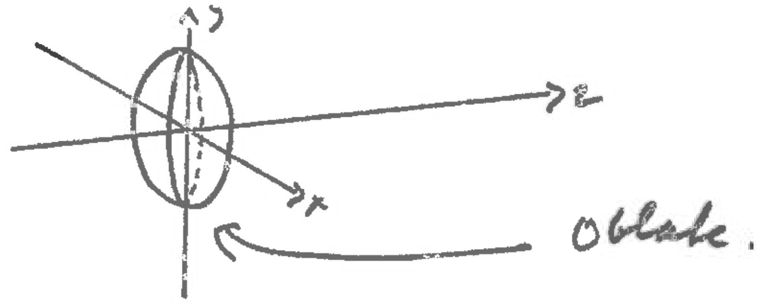
$$\frac{d\sigma}{d\Omega} = \left(\frac{m\alpha\sqrt{\pi}}{2\hbar^2}\right)^2 e^{-\frac{k^2}{2}\left(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2(1-\cos\theta)^2\right)}$$

b)

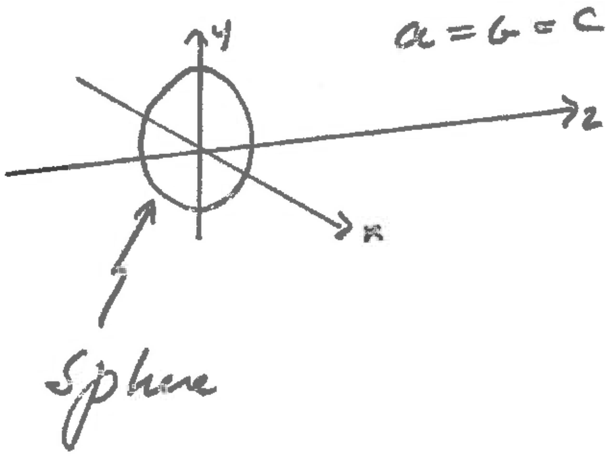
$a > b > c$



$a = b > c$



$a = b = c$



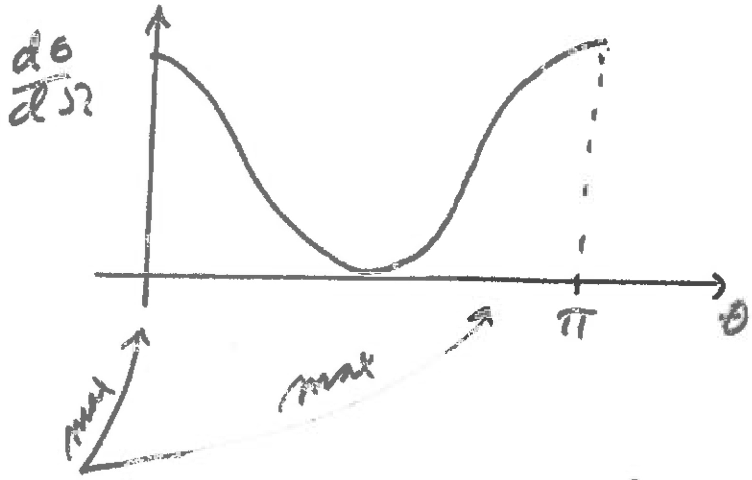
c) When $a > b > c$, the scattering has no symmetries.

When $a = b > c$, the ϕ-dependence disappears:

$$\frac{d\sigma}{d\Omega} = \left(\frac{m\alpha\sqrt{\pi}}{2\hbar^2}\right)^2 e^{-\frac{k^2}{2}\left(a^2\sin^2\theta + c^2(1-\cos\theta)^2\right)}$$

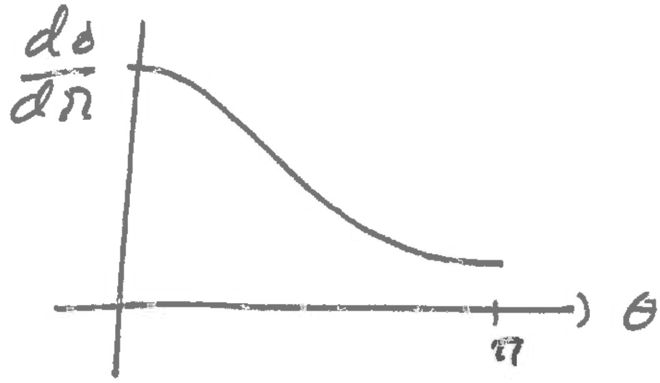
Suppose now $a \gg c$:

$$\frac{d\sigma}{d\Omega} \propto e^{-\frac{k^2}{2}a^2\sin^2\theta}$$



This is characteristic of a pancake disk.

Suppose now $a = b = c$:



This is typical of scattering from a spherical repulsive potential with finite range.

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Professor Alex Hansen

Karakterregel: Bokstavkarakterer

Kandnr	Resultat
10000	A
10002	B
10003	E
10004	A
10006	E
10007	C
10008	E
10011	B
10012	E
10013	C
10015	B
10016	C
10017	D
10018	E
10019	F
10020	C
10026	E
10028	C
10030	A

Antall kandidater til sensur i TFY4205 S1 2013 12: 19

2/1/14 Alex Hansen

Dato Sensorenes underskrifter