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TFY 4205 Quantum Mechanics II

Solution set for final exam, 20.12.2014

Problem 1

a) $$\underline{\underline{-[\xi,\eta]}} = \left(\frac{1}{eB}\right)^2 \Big[(p_y + e\underbrace{A_y}_{=0})(p_x + e\underbrace{A_x}_{=-eBy}) - (p_x + e\underbrace{A_x}_{=-eBy})(p_y + e\underbrace{A_y}_{=0}) \Big]$$

$$= \frac{1}{eB}\left(y p_y - p_y y\right) = i\frac{\hbar}{eB} = \underline{\underline{i\ell^2}}$$

$$\underline{\underline{[X,Y]}} = [x-\xi, y-\eta] = \underbrace{[x,y]}_{=0} - \underbrace{[x,\eta]}_{=+i\ell^2} - \underbrace{[\xi,y]}_{=+i\ell^2} + \underbrace{[\xi,\eta]}_{=-i\ell^2} = \underline{\underline{+i\ell^2}}$$

$$\underline{\underline{[\xi,X]}} = [\xi, x-\xi] = [\xi,x] = \underline{\underline{0}}$$

$$\underline{\underline{[\eta,Y]}} = [\eta, y-\eta] = [\eta,y] = \underline{\underline{0}}$$

$$\underline{\underline{[\xi,Y]}} = [\xi, y-\eta] = [\xi,y] - [\xi,\eta] = \underline{\underline{0}}$$

$$\underline{\underline{[\eta,X]}} = [\eta, x-\xi] = [\eta,x] - [\eta,\xi] = \underline{\underline{0}}$$

b) $\underline{\underline{H}} = \frac{1}{2m}(\vec{p} + e\vec{A})^2$

$$= \frac{1}{2m}\left[(p_x + eA_x)^2 + (p_y + eA_y)^2\right]$$

$$= \frac{(eB)^2}{2m}\left[\left(\frac{1}{eB}\right)^2(p_x + eA_x)^2 + \left(-\frac{1}{eB}\right)^2(p_y + eA_y)^2\right]$$

$$= \frac{m}{2}\frac{(eB)^2}{m^2}\left[\xi^2 + \eta^2\right] = \underline{\underline{\frac{m}{2}\omega^2(\xi^2 + \eta^2)}}$$

We have $[\eta, \xi] = i l^2 \rightarrow$

$$[\eta, (eB\xi)] = i\hbar$$

We define $\pi = eB\xi \Rightarrow$

$$\underline{[\eta, \pi] = i\hbar}$$

Hamiltonian: $\underline{\underline{H}} = \frac{m}{2}\omega^2(\xi^2 + \eta^2)$

$$= \underline{\underline{\frac{1}{2m}\pi^2 + \frac{m}{2}\omega^2\eta^2}}$$

This is the harmonic oscillator hamiltonian with coordinate η and momentum π.

The electron performs a circular motion around the guiding center coordinates. Due to the commutators, one direction

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acts as a coordinate, whereas the orthogonal one acts as a momentum.

c) $\underline{\underline{[X, H]}} = \frac{m}{2}\omega^2 [X, \xi^2 + \eta^2] \underline{\underline{= 0}}$

Since $[X, \xi] = [X, \eta] = 0$

Same argument for $\underline{\underline{[Y, H] = 0}}$

The energy eigenfunctions cannot be eigenfunctions of both X and Y at the same time since $[X, Y] \neq 0$.

Since $[\eta, \xi] = i\ell^2 \neq 0$ and $[X, Y] = i\ell^2 \neq 0$ – and all cross-commutators between ξ, η and X, Y are zero, we may use one variable from each group to construct the energy eigenfunctions, but not all four at the same time.

We have the commutator $[X, Y] = i\ell^2$ or $[X, (eBY)] = i\hbar$. Hence, we may represent the operator Y as

$$(eBY) = -i\hbar \frac{\partial}{\partial x} \quad \text{or}$$

$$Y = -i\ell^2 \frac{\partial}{\partial x}$$

With $\psi_{pn}(X,Y) = \frac{e^{i\Upsilon X/\ell^2}}{\sqrt{L}}\,\phi_n(Y)$,

we have that

$$\underline{Y\,\psi_{pn}(X,Y)} = -i\ell^2\frac{\partial}{\partial X}\psi_{pn}(X,Y)$$

$$\underline{\underline{= \Upsilon\,\psi_{pn}(X,Y)}}$$

Periodicity in X-direction: $e^{i\Upsilon L/\ell^2} = 1$

$$\Rightarrow \Upsilon L/\ell^2 = 2\pi k$$

$$\Rightarrow \underline{\underline{\Upsilon = \frac{2\pi\ell^2}{L}k.}}$$

In Y-direction: $0 \le \Upsilon \le W \Rightarrow \underline{\underline{0 \le k \le \frac{LW}{2\pi\ell^2}}}$

Degeneration: $\frac{LW}{2\pi\ell^2} = \frac{LWeB}{2\pi\hbar} = \underline{\underline{\frac{e\Phi_{tot}}{h}}}$

d) Number of electrons in system: $\rho \cdot WL$.
For them all to fit in the first Landau level:

$$\rho WL = 2\,\frac{\Phi_{tot}}{h/e} = 2\,\frac{WLB_0}{h/e}$$

$$\Rightarrow \underline{\underline{B_0 = \frac{\rho}{2}\frac{h}{e}}}$$

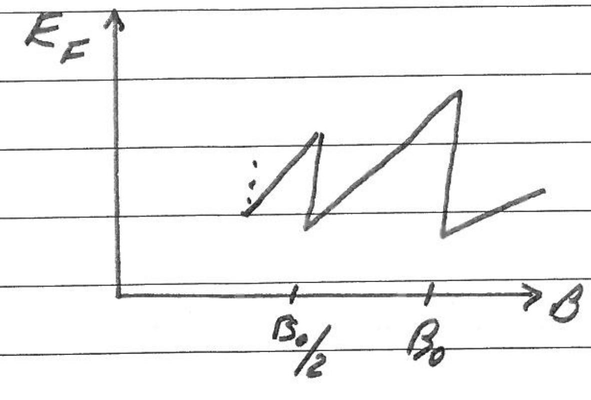
For $B > B_0$: All electrons in lowest Landau level ($n=0$). The Fermi

energy is then $E_F = \frac{1}{2} e\hbar B/m$.
Since the energy of the Landau levels is given by $(n+\frac{1}{2})\hbar eB/m$.

When $\frac{1}{2}B_0 < B < B_0$, the electrons that no longer fit into the lowest Landau level, occupy the next one $(n=1)$. The Fermi level then becomes $E_F = \frac{3}{2} e\hbar B/m$.

Every time the strength of the magnetic field reaches a value B_0/K where K is an integer, a new Landau level is occupied, and the Fermi energy jumps.

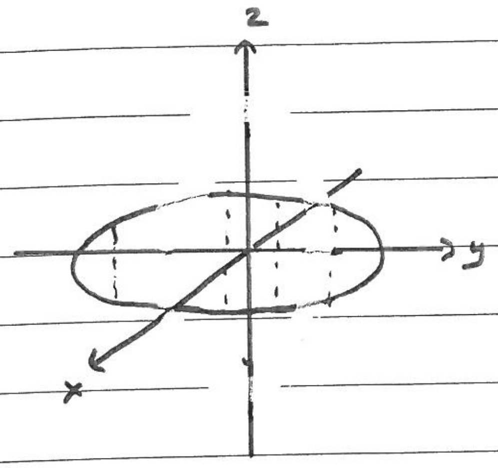
The result is



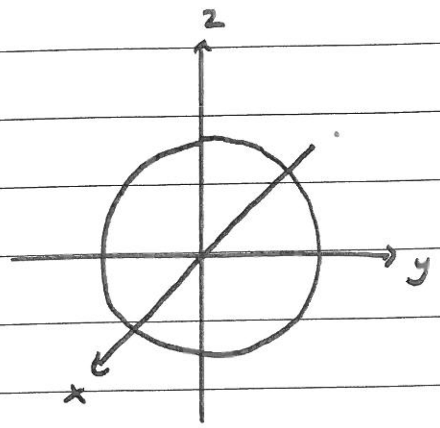
See discussion in Hemmer's book, p. 306.

Problem 2

a)



$b > c$



$b = c$

b)

$$\underline{\int d^3\vec{r}\, V(r)\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}}$$

$$= \int dx\, dy\, dz \quad \frac{\alpha}{\pi b c}\, e^{-\left(\frac{y}{b}\right)^2 + i(k_y - k_y')\cdot y}$$

$$e^{-\left(\frac{z}{c}\right)^2 + i(k_z - k_z')\cdot z} \quad e^{i(k_x - k_x')\cdot x} \quad \delta(x)$$

$$= \frac{\alpha}{\pi b c} \left[\int_{-\infty}^{+\infty} dy\, e^{-\left(\frac{y}{b}\right)^2 + i(k_y - k_y')\cdot y} \right]$$

$$\left[\int_{-\infty}^{+\infty} dz\, e^{-\left(\frac{z}{c}\right)^2 + i(k_z - k_z')\cdot z} \right]$$

$$= \frac{\alpha}{\pi b c}\left[\sqrt{\pi}\, b\, e^{-\left(\frac{b}{2}\right)^2 (k_y - k_y')^2}\right]$$

$$\left[\sqrt{\pi}\, c\, e^{-\left(\frac{c}{2}\right)^2 (k_z - k_z')^2}\right]$$

$$\underline{= \alpha\, e^{-\left(\frac{b}{2}\right)^2 (k_y - k_y')^2 - \left(\frac{c}{2}\right)^2 (k_z - k_z')^2}}$$

$$q_y = k_y - k_y', \quad q_z = k_z - k_z'$$

$$f^B = -\frac{m\alpha}{2\pi\hbar^2}\, e^{-\frac{b^2}{4} q_y^2 - \frac{c^2}{4} q_z^2}$$

$$\begin{cases} q_y = k \sin\theta \sin\phi \\ q_z = k(1-\cos\theta) \end{cases}$$

$$\underline{\underline{\frac{d\sigma}{d\theta} = |f^B|^2 = \left(\frac{m\alpha}{2\pi\hbar^2}\right)^2 e^{-\frac{k^2}{2}\left(b^2 \sin^2\theta \sin^2\phi + c^2(1-\cos\theta)^2\right)}}}$$