TFY 4205 Quantum Mechanics II

Solution Set, Exam August 8, 2015

Problem 1

a) Potential energy:

$$V = mgl(1-\cos\theta) \simeq \frac{1}{2} mgl\,\theta^2$$

for small θ.

Kinetic energy:

$$T = \frac{1}{2} m l^2 \left(\frac{d\theta}{dt}\right)^2$$

The Hamiltonian:

$$H = T + V = \frac{1}{2} m l^2 \left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} mgl\,\theta^2$$
$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m \frac{g}{l} x^2, \quad x = l\theta$$

By comparing with the one-dimensional harmonic oscillator, we find the energy spectrum

$$\underline{\underline{E_n = \left(n+\frac{1}{2}\right)\hbar\omega}}$$

where $\underline{\underline{\omega = \sqrt{\frac{g}{l}}}}$

b) The perturbation is

$$H' = mgl(1-\cos\theta) - \frac{1}{2} mgl\,\theta^2$$

$$= -\frac{1}{24} mgl\,\theta^4 = -\frac{1}{24}\frac{mg}{l^3}x^4$$

The ground state wave function for the harmonic oscillator is

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

The lowest order correction to the ground state is

$$E' = \langle 0|H'|0\rangle = -\frac{1}{24}\frac{mg}{l^3}\langle 0|x^4|0\rangle$$

where

$$\langle 0|x^4|0\rangle = \left(\frac{m\omega}{\hbar\omega}\right)^{1/2}\int_{-\infty}^{+\infty} dx\, x^4 e^{-\frac{m\omega}{\hbar}x^2}$$

$$= \frac{3}{4}\left(\frac{m\omega}{\hbar\pi}\right)^{-1}$$

Hence,

$$E' = -\frac{\hbar^2}{32ml^2}$$

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Problem 2

a)

The Schrödinger equation is

$$\frac{1}{2m}(\vec{p}-q\vec{A})^2\psi(\vec{r}) = E\psi(\vec{r}).$$

We make the transformation

$$\begin{cases} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f \\ \psi \to \psi' = \psi e^{iqf/\hbar} \end{cases}$$

We find

$$(\vec{p}-q\vec{A}')\psi' = e^{iqf/\hbar}(\vec{p}-q\vec{A})\psi$$

and

$$(\vec{p}-q\vec{A}')^2\psi' = (\vec{p}-q\vec{A}')e^{iqf/\hbar}(\vec{p}-q\vec{A})\psi$$

$$= e^{iqf/\hbar}(\vec{p}-q\vec{A})^2\psi$$

We have here used that

$$\vec{p}\psi' = \frac{\hbar}{i}\vec{\nabla}(e^{iqf/\hbar}\psi) = e^{iqf/\hbar}(q\vec{\nabla}f + \vec{p})\psi$$

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Hence, we find

$$\frac{1}{2m}(\vec{p}-q\vec{A}')^2\psi' = E\psi'$$

This shows that under the gauge transformation $\vec{A}' = \vec{A} + \vec{\nabla} f$ the Schrödinger equation remains unchanged and there is only a phase difference between the original and the transformed wave function. The system is gauge invariant.

b) The Hamiltonian is now

$$H = \frac{1}{2m}\left[(p_x + qBy)^2 + p_y^2 + p_z^2\right]$$

Since $[p_x, H] = [p_z, H] = 0$ as H does not explicitly depend on x or z, we may choose the complete set of mechanical variables (p_x, p_z). The corresponding eigenstate is

$$\psi(x,y,z) = e^{i(p_x x + p_z z)/\hbar}\chi(y)$$

Substituting into the Schrödinger equation, we get

(5)

$$\frac{1}{2m}\left[(p_x + qBy)^2 - \hbar^2 \partial_y^2 + p_z^2 \right] X(y) = EX$$

We define

$$p_x / qB = -y_0 .$$

The Schrödinger equation now becomes

$$-\frac{\hbar^2}{2m} X'' + \frac{m}{2}\left(\frac{qB}{m}\right)^2 (y-y_0)^2 X$$

$$= \left(E - \frac{p_z^2}{2m}\right) X .$$

This is the equation of motion for a harmonic oscillator. Hence, the energy levels are

$$E = \frac{\hbar^2 k_z^2}{2m} + \left(n + \frac{1}{2}\right) \hbar \frac{|q|}{m}$$

where $n = 0, 1, 2, \ldots$, $k_z = p_z / \hbar$ and the wave functions are

$$\psi_{p_x, p_z, n}(x, y, z) = e^{i(p_x x + p_z z)/\hbar} X_n (y - y_0)$$

where X_n is the nth eigenstate of the harmonic oscillator.

Problem 3

a) $\dfrac{d\sigma}{d\Omega} = \dfrac{\text{\# particles scattered into } d\Omega \text{ per unit time}}{d\Omega\,(\text{\# of incoming particles per area per time})}$.

b) The first Born approximation may be found by using the incoming wave $e^{i\vec{k}_i\cdot\vec{r}}$ for $\psi(\vec{r})$.

Hence, we have

$$f^B(\theta,\varphi) = -\frac{1}{4\pi}\int d^3r'\, e^{-i\vec{k}_f\cdot\vec{r}\,'}\, U(\vec{r}\,')\, e^{i\vec{k}_i\cdot\vec{r}\,'}$$

$$= -\frac{1}{4\pi}\int d^3r'\, e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}\,'}\, U(\vec{r}\,')$$

$$= -\frac{1}{4\pi}\int d^3r'\, e^{-i\vec{q}\cdot\vec{r}\,'}\, U(\vec{r}\,').$$

We then assume that $V(\vec{r}) = V(r)$ so that $U(\vec{r}) = U(r)$. This gives

$$f^B(\theta,\varphi) = -\frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \int_0^{\infty} dr'\, r'^2\, e^{i\vec{q}\cdot\vec{r}\,'}\, U(r').$$

We orient the z-axis in parallel with $\vec{q}$. This gives

$$f^B(\theta,\varphi) = f^B(q)$$

$$= -\frac{1}{4\pi}\, 2\pi \int_0^\infty dr\, r^2\, U(r) \int_0^\pi d\theta' \sin\theta'\, e^{-iqr\cos\theta'}$$

$$= -\frac{1}{2}\int_0^\infty dr\, r^2\, U(r) \left| \frac{e^{iqr\cos\theta'}}{iqr} \right|_0^\pi$$

$$= -\frac{1}{2}\int_0^\infty dr\, r^2\, U(r)\, \frac{1}{qr} \left\{ \frac{e^{iqr} - e^{-iqr}}{i} \right\}$$

$$= -\frac{1}{q}\int_0^\infty dr\, r\, U(r) \sin qr$$

$$= -\frac{2m}{\hbar^2 q}\int_0^\infty dr\, r \sin qr\, V(r)$$

c) From b) we have

$$f^B(q) = -\frac{2m}{\hbar^2 q}\int_0^a dr\, r \sin(qr)\, V_0$$

$$= -\frac{2mV_0}{\hbar^2 q^3}\left(\sin(qa) - qa\cos(qa)\right),$$

where $q = 2k\sin(\theta/2)$

Hence,

$$\frac{d\sigma^B}{d\Omega} = |f^B(q)|^2 = \frac{2mV_0}{\hbar^2 q^3}\left(\sin(qa) - qa\cos(qa)\right)^2.$$