

Institutt for fysikk

Eksamensoppgave i TFY4210 Kvanteteorien for mangepartikkelsystemer

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Godkjent kalkulator

Rottmann: Matematisk Formelsamling

Rottmann: Mathematische Formelsammlung

Barnett & Cronin: Mathematical Formulae

Annen informasjon:

Eksamenen har 3 oppgaver. Noen formler kan finnes på siste side.

Oppgavene ble utarbeidet av John Ove Fjærestad og drøftet med Jens Oluf Andersen.

Målform/språk: Bokmål

Antall sider (inkludert forside og vedlegg): 7

Kontrollert av:

Dato

Sign

Oppgave 1

Se på Hamiltonoperatoren

$$H = \varepsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta(a_1^\dagger a_2^\dagger + \text{h.c.}) \quad (1)$$

der $\varepsilon > 0$ og $\Delta > 0$ er parametre, og a_1 og a_2 er fermioniske operatører som tilfredsstiller kanoniske antikommuteringsrelasjoner. Dvs. $\{a_i, a_j^\dagger\} = \delta_{ij}$ ($i, j = 1, 2$), og alle andre antikommutatorer som involverer disse operatorene er null.

For å skrive Hamiltonoperatoren på diagonal form transformerer vi til et nytt sett c_1, c_2 av fermioniske operatører. Transformasjonen er

$$a_1 = uc_1 - vc_2^\dagger, \quad (2)$$

$$a_2 = uc_2 + vc_1^\dagger, \quad (3)$$

der u og v er reelle tall.

(a) Bruk kravet om at c -operatorene også skal tilfredsstille kanoniske antikommuteringsrelasjoner til å vise at

$$u^2 + v^2 = 1 \quad (4)$$

(for å vise dette, er det tilstrekkelig at du regner ut kun én utvalgt antikommutator). Dette resultatet kan brukes til å skrive $u = \cos \theta$, $v = \sin \theta$, der θ er en vinkel.

(b) Vis at ved å velge θ slik at

$$\tan 2\theta = \frac{\Delta}{\varepsilon} \quad (5)$$

blir leddet proporsjonalt med $(c_1^\dagger c_2^\dagger + \text{h.c.})$ i H lik null.

(c) Vis at med dette valget kan H skrives som

$$H = F(c_1^\dagger c_1 + c_2^\dagger c_2) + G. \quad (6)$$

Gi uttrykkene for F og G som funksjoner av ε og Δ (velg den positive løsningen for $\cos 2\theta$).

(d) Bestem energien og degenerasjonen av hvert energinivå til H . Anta her at den eneste begrensningen på partikkeltall kommer fra Pauliprinsippet, dvs. bruk det store kanoniske ensemble.

Oppgave 2

Se på en ferromagnet på et kvadratisk gitter. Hamiltonoperatoren er

$$H = - \sum_{\langle i,j \rangle} [J_{\perp}(S_i^x S_j^x + S_i^y S_j^y) + J_z S_i^z S_j^z]. \quad (7)$$

Her er $J_z > 0$ og vi antar at $0 \leq J_{\perp} \leq J_z$. Vi antar også at kun nærmeste-nabo gitterpunkt vekselvirker med hverandre, slik at summen er over alle par av nærmeste-nabo gitterpunkter (hvert slikt par telles kun en gang).

(a) Bruk spinnbølgeteori til å regne ut grunntilstandsenergien E_0 og magnon-dispersjonen $\omega_{\mathbf{k}}$ (i denne analysen skal du neglisjere ledd som beskriver vekselvirkninger mellom magnoner).

(b) Vis at for små $|\mathbf{k}|$ kan magnon-dispersjonen uttrykkes vha. to parametre Δ og m som

$$\omega_{\mathbf{k}} \approx \Delta + \frac{k^2}{2m} \quad (8)$$

og finn uttrykk for Δ og m . Her er Δ den lavere grensen av magnon-energien og kalles energigapet, mens m kan tolkes som en masse (merk at vi har satt $\hbar = 1$; med \hbar inkludert sees det m -avhengige leddet å være av velkjent kinetisk-energi form $\hbar^2 k^2 / (2m)$).

(c) Hva er verdien til Δ i grensen $J_{\perp} \rightarrow J_z$? Er dette hva du ville forvente i lys av Goldstones teorem? Forklar.

Oppgave 3

I denne oppgaven skal du se på elektroner i et uordnet potensial, som diskutert i forelesningene. Hamiltonoperatoren er

$$H = H_0 + V \quad (9)$$

der

$$H_0 = \sum_{\mathbf{k}} \xi_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} \quad (10)$$

(som i forelesningene dropper vi spinn-indeksen) og vekselvirkningene mellom elektronene og urenheterne er beskrevet av

$$V = \sum_{\mathbf{k}, \mathbf{k}'} U(\mathbf{k}' - \mathbf{k}) \rho(\mathbf{k}' - \mathbf{k}) \hat{c}_{\mathbf{k}'}^{\dagger} \hat{c}_{\mathbf{k}}, \quad (11)$$

der

$$U(\mathbf{k}) = \frac{1}{\Omega} \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} U(\mathbf{r}), \quad (12)$$

$$\rho(\mathbf{k}) = \sum_{j=1}^N e^{-i\mathbf{k}\cdot\mathbf{R}_j}, \quad (13)$$

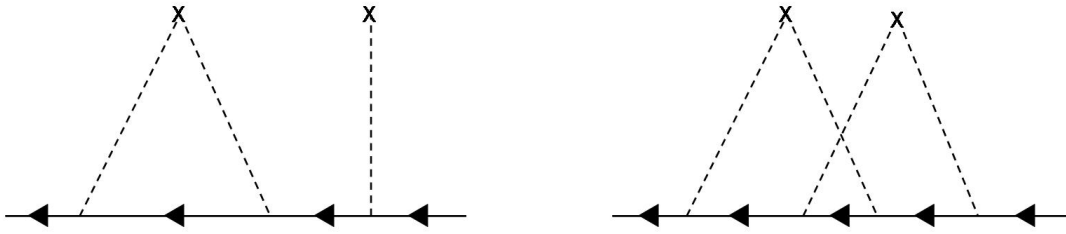
der Ω er systemets volum, $U(\mathbf{r} - \mathbf{R}_j)$ er potensialet et elektron med posisjon \mathbf{r} føler pga. urenhet nr. j med posisjon \mathbf{R}_j , og N er antallet urenheter. I forelesningene diskuterte vi en perturbasjonsutvikling for én-partikkel Matsubara Green-funksjonen $\mathcal{G}(\mathbf{k}, \mathbf{k}'; ip_m)$ der p_m er en fermionisk Matsubara-frekvens. Ved å midle over posisjonene til urenheterne, ble den resulterende Green-funksjonen \mathbf{k} -diagonal: $\bar{\mathcal{G}}(\mathbf{k}, \mathbf{k}'; ip_m) = \bar{\mathcal{G}}(\mathbf{k}, ip_m) \delta_{\mathbf{k}, \mathbf{k}'}$. Vi representerte hvert ledd i perturbasjonsutviklingen for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$ med et Feynman-diagram og etablerte Feynman-reglene for å oversette mellom diagrammene og deres assosierte matematiske uttrykk.

(a) Anta at du blir presentert et vilkårlig Feynman-diagram i perturbasjon-utviklingen for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$. Hvordan ville du identifisere

1. dets orden n (med hensyn paa urenhetspotensialet).
2. dets avhengighet av tettheten av urenheter $n_{\text{imp}} = N/\Omega$.

(b) Se på de to Feynman-diagrammene under som opptrer i perturbasjon-utviklingen for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$.

For hvert diagram:



1. Gi dets matematiske uttrykk (prøv ikke å evaluere noen summer over bølgevektorer).
2. Avgjør om diagrammet er redusibelt eller irreducibelt (grunngi konklusjonen din). Dersom diagrammet er irreducibelt, tegn det assosierte selvenergi-diagrammet og gi dets matematiske uttrykk.

(c) I forelesningene viste vi at $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$ kan uttrykkes som

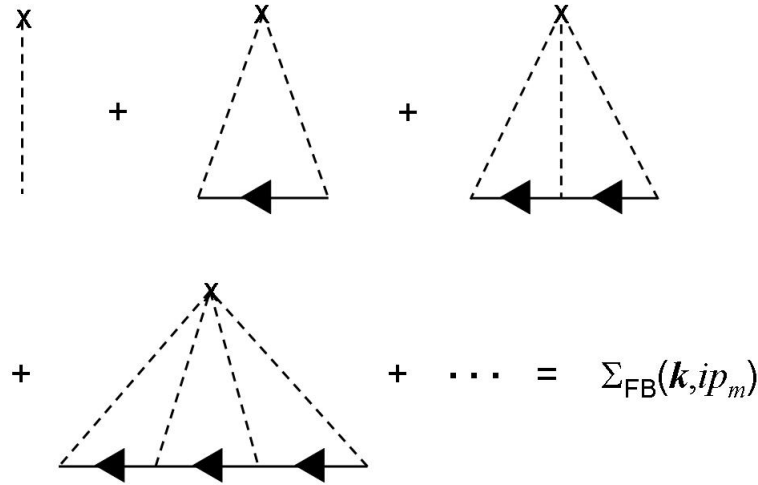
$$\bar{\mathcal{G}}(\mathbf{k}, ip_m) = \frac{1}{(\mathcal{G}^{(0)}(\mathbf{k}, ip_m))^{-1} - \Sigma(\mathbf{k}, ip_m)} = \frac{1}{ip_m - \xi_{\mathbf{k}} - \Sigma(\mathbf{k}, ip_m)} \quad (14)$$

der $\Sigma(\mathbf{k}, ip_m)$ er selvenergien, definert som summen av alle selvenergi-diagrammer. Vi fant et approksimativt resultat for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$ ved å approksimere selvenergien $\Sigma(\mathbf{k}, ip_m)$ som

$$\begin{array}{c} \text{X} \\ | \\ \vdots \end{array} + \begin{array}{c} \text{X} \\ / \quad \backslash \\ \leftarrow \end{array} = \Sigma_{1B}(\mathbf{k}, ip_m)$$

Dette kalles “første Born-approksimasjon,” som er grunnen til at vi her har definert summen av disse to selvenergi-diagrammene som $\Sigma_{1B}(\mathbf{k}, ip_m)$. I det følgende vil vi studere en annen approksimasjon for selvenergien, gitt av den uendelige summen av *alle* selvenergi-diagrammer med *ett* urenhetskryss (se figuren under).

Dette kalles “den fulle Born-approksimasjonen,” som er grunnen til at vi har definert denne summen som $\Sigma_{FB}(\mathbf{k}, ip_m)$. Merk at de to første leddene i denne uendelige summen utgjør “første Born-approksimasjon” $\Sigma_{1B}(\mathbf{k}, ip_m)$.



1. Forklar under hvilke fysiske betingelser (spesifisert av urenhets tettheten n_{imp} og styrken til spredningspotensialet $U(\mathbf{r})$) du forventer at Σ_{1B} er en god approksimasjon til den totale selvenergien. Gjør det samme for Σ_{FB} . Begrunn svarene dine.
2. Gi det matematiske uttrykket for det n 'te diagrammet i $\Sigma_{FB}(\mathbf{k}, ip_m)$, dvs. diagrammet med n vekselvirkningslinjer der n er et vilkårlig positivt heltall. (Det kan være til hjelp å først se på uttrykkene for de første få diagrammene i $\Sigma_{FB}(\mathbf{k}, ip_m)$ før du ser på tilfellet for en generell n).
3. Anta at urenhetspotensialet $U(\mathbf{r})$ har svært kort rekkevidde, slik at dets Fourier-transform $U(\mathbf{k})$ kan approksimeres med en konstant U , dvs. $U(\mathbf{k}) \equiv U$ for alle \mathbf{k} . Vis at i dette tilfellet er $\Sigma_{FB}(\mathbf{k}, ip_m)$ gitt som

$$\Sigma_{FB}(\mathbf{k}, ip_m) = \frac{NU}{1 - U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1, ip_m)} \equiv \Sigma_{FB}(ip_m). \quad (15)$$

4. I forelesningene fant vi at for et urenhetspotensial med svært kort rekkevidde er det andre diagrammet i Σ_{1B} og Σ_{FB} i figurene over gitt som

$$-\frac{i}{2\tau_{1B}} \text{sgn}(p_m) \quad \text{der} \quad \frac{1}{\tau_{1B}} = 2\pi n_{\text{imp}} u^2 D(0). \quad (16)$$

(I forelesningene ble τ_{1B} bare kalt τ .) Her er $u = U\Omega$ og $D(0)$ er en tilstandstetthetsfaktor hvis presise definisjon ikke er viktig her. Bruk resultatet (16) til å vise at

$$\text{Im} \Sigma_{FB}(ip_m) = -\frac{1}{2\tau_{FB}} \text{sgn}(p_m) \quad (17)$$

og gi et uttrykk for parameteren τ_{FB} .

Formler

Trigonometriske identiteter:

$$\cos 2x = \cos^2 x - \sin^2 x, \quad (18)$$

$$\sin 2x = 2 \sin x \cos x, \quad (19)$$

$$\cos^2 x = \frac{1}{1 + \tan^2 x} \quad (20)$$

Spinn-vekselvirkninger:

$$S_i^x S_j^x + S_i^y S_j^y = \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)$$

Holstein-Primakoff-representasjonen:

$$S_j^+ = \sqrt{2S - n_j} a_j,$$

$$S_j^- = a_j^\dagger \sqrt{2S - n_j},$$

$$S_j^z = S - n_j,$$

der $n_j \equiv a_j^\dagger a_j$

$$a_j = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j} a_{\mathbf{k}}$$

Gittersum:

$$\frac{1}{N} \sum_j e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_j} = \delta_{\mathbf{k},\mathbf{k}'}$$

Geometrisk rekke:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$



NTNU – Trondheim
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Examination paper for TFY4210 Quantum theory of many-particle systems

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Examination date: 7 June

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Permitted examination support material: C

Approved calculator

Rottmann: Matematisk Formelsamling

Rottmann: Matematische Formelsammlung

Barnett & Cronin: Mathematical Formulae

Other information:

The exam has 3 problems. Some formulas can be found on the last page.

The problems were developed by John Ove Fjærestad and discussed with Jens Oluf Andersen.

Language: English

Number of pages (including front page and attachments): 7

Checked by:

Date

Signature

Problem 1

Consider the Hamiltonian

$$H = \varepsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta(a_1^\dagger a_2^\dagger + \text{h.c.}) \quad (1)$$

where $\varepsilon > 0$ and $\Delta > 0$ are parameters, and a_1 and a_2 are fermionic operators satisfying canonical anticommutation relations. Thus $\{a_i, a_j^\dagger\} = \delta_{ij}$ ($i, j = 1, 2$), and all other anticommutators involving these operators vanish.

In order to write the Hamiltonian in diagonal form we transform to a new set c_1, c_2 of fermionic operators. The transformation reads

$$a_1 = uc_1 - vc_2^\dagger, \quad (2)$$

$$a_2 = uc_2 + vc_1^\dagger, \quad (3)$$

where u and v are real numbers.

(a) Use the requirement that the c -operators should also satisfy canonical anticommutation relations to show that

$$u^2 + v^2 = 1 \quad (4)$$

(to show this, it is sufficient that you calculate just one selected anticommutator). This result can be used to write $u = \cos \theta$, $v = \sin \theta$, where θ is an angle.

(b) Show that by choosing θ such that

$$\tan 2\theta = \frac{\Delta}{\varepsilon} \quad (5)$$

the term proportional to $(c_1^\dagger c_2^\dagger + \text{h.c.})$ in H vanishes.

(c) Show that with this choice, H can be written as

$$H = F(c_1^\dagger c_1 + c_2^\dagger c_2) + G. \quad (6)$$

Give expressions for F and G in terms of ε and Δ (pick the positive solution for $\cos 2\theta$).

(d) Determine the energy and degeneracy of each energy level of H . Here, assume that the only restriction on particle numbers comes from the Pauli principle, i.e. work in the grand canonical ensemble.

Problem 2

Consider a ferromagnet on a square lattice. The Hamiltonian is

$$H = - \sum_{\langle i,j \rangle} [J_{\perp}(S_i^x S_j^x + S_i^y S_j^y) + J_z S_i^z S_j^z]. \quad (7)$$

Here $J_z > 0$ and we will assume that $0 \leq J_{\perp} \leq J_z$. We also assume that only nearest-neighbour sites interact with each other, so the sum is over all pairs of nearest-neighbour sites (each such pair being counted only once).

(a) Use spin-wave theory to calculate the ground state energy E_0 and the magnon dispersion $\omega_{\mathbf{k}}$ (in this analysis, neglect terms describing interactions between magnons).

(b) Show that for small $|\mathbf{k}|$, the magnon dispersion can be expressed in terms of two parameters Δ and m as

$$\omega_{\mathbf{k}} \approx \Delta + \frac{k^2}{2m} \quad (8)$$

and find expressions for Δ and m . Here Δ is the lower bound on the magnon energy and is called the energy gap, while m can be interpreted as a mass (note that we have set $\hbar = 1$; with \hbar reinstated, the term involving m is seen to be of the familiar kinetic energy form $\hbar^2 k^2 / (2m)$).

(c) What is the value of Δ in the limit $J_{\perp} \rightarrow J_z$? Is this what you would expect in view of Goldstone's theorem? Explain.

Problem 3

In this problem you will consider electrons in a disordered potential, as discussed in the lectures. The Hamiltonian is given by

$$H = H_0 + V \quad (9)$$

where

$$H_0 = \sum_{\mathbf{k}} \xi_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} \quad (10)$$

(we drop the spin index as in the lectures) and the interaction between the electrons and the impurities is described by

$$V = \sum_{\mathbf{k}, \mathbf{k}'} U(\mathbf{k}' - \mathbf{k}) \rho(\mathbf{k}' - \mathbf{k}) \hat{c}_{\mathbf{k}'}^{\dagger} \hat{c}_{\mathbf{k}}, \quad (11)$$

where

$$U(\mathbf{k}) = \frac{1}{\Omega} \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} U(\mathbf{r}), \quad (12)$$

$$\rho(\mathbf{k}) = \sum_{j=1}^N e^{-i\mathbf{k}\cdot\mathbf{R}_j}, \quad (13)$$

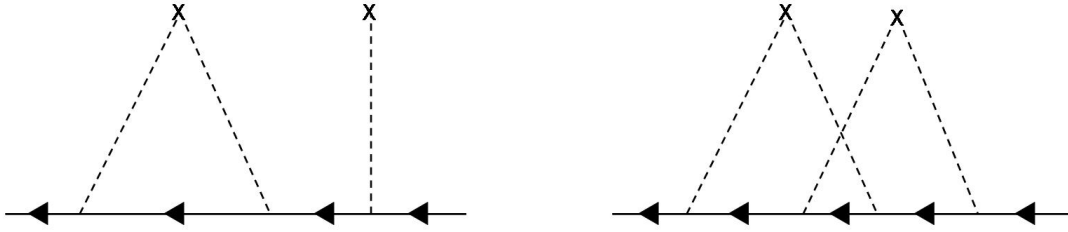
where Ω is the volume of the system, $U(\mathbf{r} - \mathbf{R}_j)$ is the potential an electron at position \mathbf{r} experiences due to the j 'th impurity at position \mathbf{R}_j , and N is the number of impurities. In the lectures we developed a perturbation expansion for the single-particle Matsubara Green function $\mathcal{G}(\mathbf{k}, \mathbf{k}'; ip_m)$ where p_m is a fermionic Matsubara frequency. Upon averaging over the positions of the impurities, the resulting Green function became \mathbf{k} -diagonal: $\bar{\mathcal{G}}(\mathbf{k}, \mathbf{k}'; ip_m) = \bar{\mathcal{G}}(\mathbf{k}, ip_m) \delta_{\mathbf{k}, \mathbf{k}'}$. We represented each term in the perturbation expansion for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$ by a Feynman diagram and established the Feynman rules for translating between the diagrams and their associated mathematical expressions.

(a) Suppose that you are presented with an arbitrary Feynman diagram in the perturbation expansion for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$. How would you identify

1. its order n (with respect to the impurity potential).
2. its dependence on the density of impurities $n_{\text{imp}} = N/\Omega$.

(b) Consider the two Feynman diagrams below that appear in the perturbation expansion for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$.

For each diagram:



1. Give its mathematical expression (do not attempt to evaluate any wavevector sums).
2. Determine whether the diagram is reducible or irreducible (justify your conclusion). If the diagram is irreducible, draw the corresponding self-energy diagram and give its mathematical expression.

(c) In the lectures we showed that $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$ can be expressed as

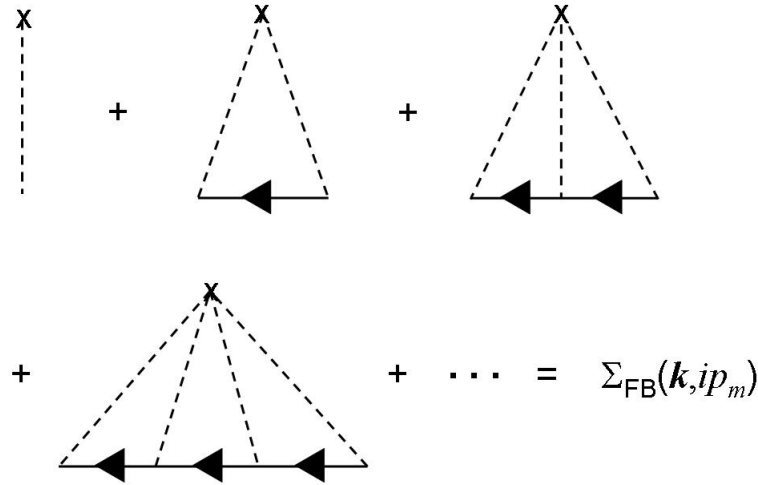
$$\bar{\mathcal{G}}(\mathbf{k}, ip_m) = \frac{1}{(\mathcal{G}^{(0)}(\mathbf{k}, ip_m))^{-1} - \Sigma(\mathbf{k}, ip_m)} = \frac{1}{ip_m - \xi_{\mathbf{k}} - \Sigma(\mathbf{k}, ip_m)} \quad (14)$$

where $\Sigma(\mathbf{k}, ip_m)$ is the self-energy, defined as the sum of all self-energy diagrams. We found an approximate result for $\bar{\mathcal{G}}(\mathbf{k}, ip_m)$ by approximating the self-energy $\Sigma(\mathbf{k}, ip_m)$ as

$$\begin{array}{c} \text{X} \\ | \\ \vdots \end{array} + \begin{array}{c} \text{X} \\ / \quad \backslash \\ \leftarrow \end{array} = \Sigma_{1\text{B}}(\mathbf{k}, ip_m)$$

This is called the "first Born approximation," which is why we have here defined the sum of these two self-energy diagrams as $\Sigma_{1\text{B}}(\mathbf{k}, ip_m)$. In the following we will study a different approximation to the self-energy, given by the infinite sum of *all* self-energy diagrams with a *single* impurity cross (see the figure below).

This is called the "full Born approximation," which is why we have defined this sum as $\Sigma_{\text{FB}}(\mathbf{k}, ip_m)$. Note that the first two terms in this infinite sum constitute the "first Born approximation" $\Sigma_{1\text{B}}(\mathbf{k}, ip_m)$.



1. Explain under what physical conditions (specified by the impurity density n_{imp} and the strength of the scattering potential $U(\mathbf{r})$) you expect $\Sigma_{1\text{B}}$ to be a good approximation to the full self-energy. Do the same for Σ_{FB} . Justify your answers.
2. Give the mathematical expression for the n th diagram in $\Sigma_{\text{FB}}(\mathbf{k}, ip_m)$, i.e. the diagram with n interaction lines where n is an arbitrary positive integer. (You may find it helpful to first consider the expressions for the first few diagrams in $\Sigma_{\text{FB}}(\mathbf{k}, ip_m)$ before you consider the case of a general n .)
3. Assume that the impurity potential $U(\mathbf{r})$ is very short-ranged, so that its Fourier transform $U(\mathbf{k})$ can be approximated by a constant U , i.e. $U(\mathbf{k}) \equiv U$ for all \mathbf{k} . Show that in this case, $\Sigma_{\text{FB}}(\mathbf{k}, ip_m)$ is given by

$$\Sigma_{\text{FB}}(\mathbf{k}, ip_m) = \frac{NU}{1 - U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1, ip_m)} \equiv \Sigma_{\text{FB}}(ip_m). \quad (15)$$

4. In the lectures we found that for a very short-ranged impurity potential, the second diagram in $\Sigma_{1\text{B}}$ and Σ_{FB} in the figures above is given by

$$-\frac{i}{2\tau_{1\text{B}}} \text{sgn}(p_m) \quad \text{where} \quad \frac{1}{\tau_{1\text{B}}} = 2\pi n_{\text{imp}} u^2 D(0). \quad (16)$$

(In the lectures $\tau_{1\text{B}}$ was just called τ .) Here $u = U\Omega$ and $D(0)$ is a density-of-states factor whose precise definition is unimportant here. Use the result (16) to show that

$$\text{Im} \Sigma_{\text{FB}}(ip_m) = -\frac{1}{2\tau_{\text{FB}}} \text{sgn}(p_m) \quad (17)$$

and give an expression for the parameter τ_{FB} .

Formulas

Trigonometric identities:

$$\cos 2x = \cos^2 x - \sin^2 x, \quad (18)$$

$$\sin 2x = 2 \sin x \cos x, \quad (19)$$

$$\cos^2 x = \frac{1}{1 + \tan^2 x} \quad (20)$$

Spin interactions:

$$S_i^x S_j^x + S_i^y S_j^y = \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)$$

Holstein-Primakoff representation:

$$S_j^+ = \sqrt{2S - n_j} a_j,$$

$$S_j^- = a_j^\dagger \sqrt{2S - n_j},$$

$$S_j^z = S - n_j,$$

where $n_j \equiv a_j^\dagger a_j$

$$a_j = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j} a_{\mathbf{k}}$$

Lattice sum:

$$\frac{1}{N} \sum_j e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_j} = \delta_{\mathbf{k},\mathbf{k}'}$$

Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$