NTNU
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Department of Physics

Examination paper for **TFY4210 Quantum theory of solids**

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Examination date: June 1 2022

Examination time (from–to): 09:00–13:00

Permitted examination support material: B: All printed and handwritten material. Approved calculator.

Other information:
If a question is unclear/vague – make your own assumptions and specify in your answer the premises you have made. Only contact academic contact in case of errors or insufficiencies in the question set.
Cheating/Plagiarism: The exam is an individual, independent work. All submitted answers will be subject to plagiarism control.
The number of points that can be scored on each question is given in the problem set. Maximum number of points is 100.
If you use results from lecture notes or other printed material, cite your sources.
If you wish to submit a blank test/withdraw from the exam, go to the menu and click "Submit blank". This cannot be undone, even if the test is still open.

Technical support during examination: Orakel support service. Phone: 73591600

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☒	

Date Signature

Problem 1 (Points: 5+10+10+10+10+5=50)

An antiferromagnetic insulator on a $2D$ square lattice may be modelled by a Hamiltonian with nearest neighbor exchange interaction J

$$H_{\text{AFMI}} = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - K \sum_i S_{iz}^2$$

Here, $J < 0$, and $K > 0$. The operators $\mathbf{S}_i$ are spin-operators, and $\langle i, j \rangle$ means that the lattice sites i and j are nearest neighbors.

a) Give a physical interpretation of the K-term in the Hamiltonian.

b) Use the Holstein-Primakoff transformation used in class for this case to find the excitation-spectrum for the magnons of the model given above, calculating to quadratic order in magnon-operators. Sketch the spectrum for small momenta. (The case $K = 0$ is solved in the lecture notes. A solution for the case $K = 0$ only will therefore not receive any ponts, but you can use the $K = 0$-solution to aid you in finding the $K \neq 0$-solution).

c) We now add an external uniform magnetic field h to the system, such that the Hamiltonian is given by

$$H_{\text{AFMI}} = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - K \sum_i S_{iz}^2 - h \sum_i S_{iz}$$

with $h > 0$. Express the added terms in Fourier-space using the magnon-operators that diagonalize the problem for $h = 0$.

d) Compute the magnon-spectra for $h > 0$.

e) For $K > 0, h > 0$, find the maximum value of $h > 0$ for which the magnon-spectra are meaningful.

f) Give a physical interpretation of what happens when h exceeds this limit. (This question can be answered even without having obtained a detailed answer in **e)**).

Problem 2 (Points: 10+10+10+10+10=50)

A model of bosonic particles on a $2D$ square lattice with N_L lattice points, is given by

$$H = -t \sum_{\langle i,j \rangle} a_i^\dagger a_j + \frac{U}{2} \sum_i n_i(n_i - 1)$$

where $n_i = a_i^\dagger a_i$ and the a-operators are bosonic creation and destruction operators. We have $(t, U) > 0$, and $\langle i, j \rangle$ indicates that lattice sites i and j are nearest neighbors.

a) Explain what the various terms in the Hamiltonian describe.

b) Introduce Fourier-transformed operators

$$a_k = \frac{1}{\sqrt{N_L}} \sum_i a_i \ \exp(i\mathbf{k} \cdot \mathbf{r}_i)$$

where $\mathbf{r}_i$ is the position of lattice point i. Show that the Hamiltonian may be written on the form

$$H = \sum_k \varepsilon_k a_k^\dagger a_k + \frac{1}{2} \sum_{k_1,\ldots,k_4} U_{k_1 k_2 k_3 k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \delta_{k_1+k_2,k_3+k_4}$$

where $\delta_{k,k'}$ is a Kronecker-delta, and give expressions for ε_k and $U_{k_1 k_2 k_3 k_4}$. Give a physical interpretation of the constraint on the summations over $k_1, ..k_4$.

c) This system may undergo Bose condensation at sufficiently low temperatures, whereby a macroscopic number of particles occupy the lowest possible energy state. Let the total number of particles be N, and let the number of particles in the lowest possible energy state be N_0. Give an expression for the condensate fraction N_0/N, stating the conditions under which the expression applies.

d) Compute the temperature dependence of the correction to the condensate fraction, at low temperatures in d dimensions.

e) What is the lowest dimension one can have Bose-condensation at $T > 0$ in the Bose Hubbard model with $U > 0$?

$$\int_0^\infty dx \frac{x^\alpha}{e^x - 1} = \zeta(\alpha + 1)\Gamma(\alpha + 1)$$

where ζ, Γ are the Riemann zeta-function and Gamma-function, respectively.