Solution sketch, Exam TFY4210 Quantum theory of many-particle systems

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Problem 1

(a) Dirac wanted a relativistic equation for the electron. The Klein-Gordon equation had originally been suggested for this, but it led to interpretational problems ("negative probabilities") due to being second order in the time derivative. (Besides, it turned out to be unable to describe particles with spin.) Dirac therefore considered an equation that was first order in the time derivative (like the Schrödinger equation), and thus had to be first order in space derivatives too, in order for space and time coordinates to appear in a similar way, as required by relativistic considerations. This led to a Hamiltonian that was a linearization of the kinetic energy operator for a free relativistic particle.

(b) We start from the linearization

$$\sqrt{\vec{p}^2+m^2} = \vec{\alpha}\cdot\vec{p}+\beta m = \alpha_1 p_x + \beta m \tag{1}$$

where $\vec{p}$ is the momentum operator and p_x its only component (one spatial dimension). Squaring both sides gives

$$p_x^2+m^2 = (\alpha_1 p_x+\beta m)^2 = \alpha_1^2 p_x^2 + (\alpha_1\beta+\beta\alpha_1)p_x m + \beta^2 m^2. \tag{2}$$

Comparing the leftmost and rightmost expressions then gives

$$\alpha_1^2 = \beta^2 = 1, \tag{3}$$

$$\alpha_1\beta+\beta\alpha_1 = 0. \tag{4}$$

(c) Using the explicit form of the Pauli matrices (given in the Formula section) and of the momentum operator gives

$$H = \sigma_3 p_x + \sigma_1 m = \begin{pmatrix} -i\partial_x & m \\ m & i\partial_x \end{pmatrix}. \tag{5}$$

We assume that the eigenstates of H take a plane-wave form,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} e^{ipx} \tag{6}$$

where ψ_1 and ψ_2 are constants (i.e. independent of x). The eigenvalue equation $H\psi = E\psi$ then gives, upon carrying out the differentations and rearranging,

$$\begin{pmatrix} p-E & m \\ m & -p-E \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{7}$$

This equation has solutions if the determinant of the matrix vanishes, i.e. if

$$0 = -(p-E)(p+E) - m^2 = -p^2 + E^2 - m^2, \tag{8}$$

which gives $E = \pm\sqrt{p^2+m^2}$.

(d) The simplest derivation proceeds by treating ψ and $\bar{\psi}$ as independent fields, and considering the Euler-Lagrange equation for $\bar{\psi}$, which reads

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0. \tag{9}$$

One finds

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0. \tag{10}$$

Inserting this into the Euler-Lagrange equation above gives the Dirac equation.

(e) Using the given formulas for the Pauli matrices one finds

$$\gamma^5 = \gamma^0\gamma^1 = \sigma_1(-i)\sigma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Taking the adjoint of the transformation of ψ gives

$$\psi^\dagger \to (e^{i\theta\gamma^5}\psi)^\dagger = \psi^\dagger e^{-i\theta\gamma^5}, \tag{12}$$

where we used that γ^5 is hermitian, which follows from (11). Thus $\bar{\psi}$ transforms as

$$\bar{\psi} = \psi^\dagger\gamma^0 \;\to\; \psi^\dagger e^{-i\theta\gamma^5}\gamma^0 = \psi^\dagger \sum_n \frac{(-i\theta)^n}{n!}(\gamma^5)^n\gamma^0 = \psi^\dagger\gamma^0 \sum_n \frac{(-i\theta)^n}{n!}(-1)^n(\gamma^5)^n = \psi^\dagger\gamma^0 e^{i\theta\gamma^5} = \bar{\psi}\, e^{i\theta\gamma^5}.$$

Here we used that interchanging γ^0 and γ^5 produces a minus sign, which follows from

$$\{\gamma^5, \gamma^0\} = \gamma^5\gamma^0 + \gamma^0\gamma^5 = \sigma_3\sigma_1 + \sigma_1\sigma_3 = 0, \tag{13}$$

where the last result was given in the formula section. This gives

$$\bar{\psi}\psi \;\to\; \bar{\psi}e^{2i\theta\gamma^5}\psi, \tag{14}$$
$$i\bar{\psi}\gamma^5\psi \;\to\; i\bar{\psi}e^{2i\theta\gamma^5}\gamma^5\psi. \tag{15}$$

The exponential can be written

$$e^{2i\theta\gamma^5} = \sum_{n=0}^{\infty} \frac{(2i\theta\gamma^5)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2i\theta\gamma^5)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(2i\theta\gamma^5)^{2n+1}}{(2n+1)!}. \tag{16}$$

Now we use that $(\gamma^5)^2 = I$, which is a consequence of (11). This gives $(\gamma^5)^{2n} = I$ and $(\gamma^5)^{2n+1} = \gamma^5$. Thus

$$e^{2i\theta\gamma^5} = I \sum_{n=0}^{\infty} \frac{(-1)^n(2\theta)^{2n}}{(2n)!} + i\gamma^5 \sum_{n=0}^{\infty} \frac{(-1)^n(2\theta)^{2n+1}}{(2n+1)!} = I\cos 2\theta + i\gamma^5 \sin 2\theta. \tag{17}$$

This gives

$$\begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix} \to \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix}.$$

The 2-component vector is therefore invariant if $2\theta = 2\pi n$, where n is an integer, i.e. if $\theta = n\pi$.

Problem 2

(a) Diagram (i):

$$\frac{1}{S}(-i\lambda)\int d^4z\; D_F(x-z)D_F(z-y)\underbrace{D_F(z-z)}_{=D_F(0)}. \tag{18}$$

Diagram (ii):

$$\frac{1}{S}(-i\lambda)^3 \int d^4z_1 \int d^4z_2 \int d^4z_3 D_F(x-z_1)D_F(y-z_1)D_F(z_1-z_2)D_F(z_1-z_3)(D_F(z_2-z_3))^3. \tag{19}$$

(b) Diagrams (i), (ii), and (iii) are invalid. Justification:

- In diagram (i) internal points (vertices) are connected to 3 propagator lines, but in φ^4 theory there should be 4.
- Diagram (ii) has no external points x and y, only internal points (vertices).
- Diagram (iii) is not a connected diagram.

(c) Using momentum conservation at each vertex shows that the momenta of the horizontal lines are all equal (and thus given by p, since the sum is an approximation for $\tilde{D}_F(p)_{\text{int}}$). Labeling the momentum variable in the j'th loop as p_j, the diagram with n loops is given by the expression

$$2^{-n}(-i\lambda)^n(\tilde{D}_F(p))^{n+1}\prod_{j=1}^{n}\int\frac{d^4p_j}{(2\pi)^4}\tilde{D}_F(p_j) = 2^{-n}(-i\lambda)^n(\tilde{D}_F(p))^{n+1}(D_F(0))^n, \tag{20}$$

where in the last expression we used that the n loop integrals are identical and equal to $D_F(0)$. Thus approximating $\tilde{D}_F(p)_{\text{int}}$ as the sum of the series from $n=0$ to ∞ gives

$$\tilde{D}_F(p)_{\text{int}} \approx \tilde{D}_F(p)\sum_{n=0}^{\infty}\left(\frac{-i\lambda\tilde{D}_F(p)D_F(0)}{2}\right)^n. \tag{21}$$

We recognize this as a geometric series. Evaluating it gives

$$\tilde{D}_F(p)_{\text{int}} \approx \frac{\tilde{D}_F(p)}{1-\frac{-i\lambda\tilde{D}_F(p)D_F(0)}{2}} = \frac{1}{(\tilde{D}_F(p))^{-1}+i\lambda D_F(0)/2}. \tag{22}$$

Using the expression for $\tilde{D}_F(p)$ given in the formula section, this simplifies to

$$\tilde{D}_F(p)_{\text{int}} \approx \frac{i}{p^2-(m^2+\frac{\lambda}{2}D_F(0))+i\epsilon}. \tag{23}$$

Problem 3

(a) Both terms in H have the same structure, each involving an expression of the form

$$\sum_j c^\dagger_{j\sigma}c_{j+r,\sigma} + \text{h.c.}, \tag{24}$$

where $r=1$ in the first term and $r=2$ in the second term. We introduce Fourier transformed operators as

$$c_{j\sigma} = \frac{1}{\sqrt{N}}\sum_k e^{ikj}c_{k\sigma}. \tag{25}$$

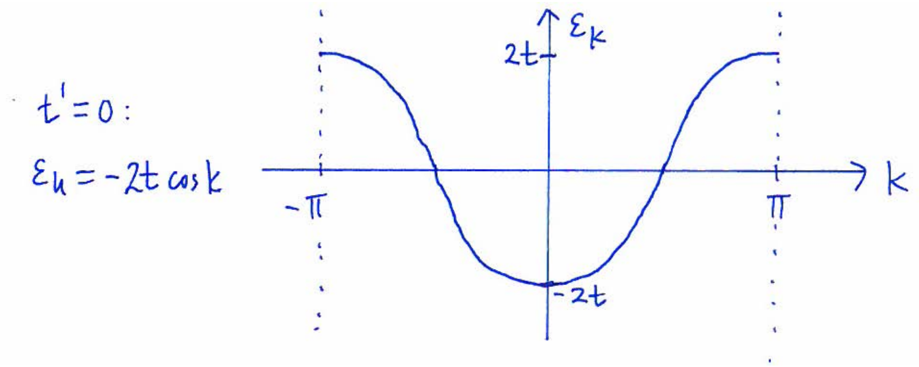
Periodic boundary conditions means $c_{j\sigma} = c_{j+N,\sigma}$ which implies $e^{ikN}=1$, i.e. $k=2\pi n/N$, where n is integer. Choosing the N values of n closest to zero ($n=-N/2,-N/2+1,\ldots,N/2-1$) then gives that the N inequivalent k vectors are in the 1st Brillouin zone, uniformly spaced by $2\pi/N$. Consider now

$$\sum_j c^\dagger_{j\sigma}c_{j+r,\sigma} = \frac{1}{N}\sum_j\sum_{k,k'}e^{-ikj}e^{ik'(j+r)}c^\dagger_{k\sigma}c_{k'\sigma} = \sum_{k,k'}c^\dagger_{k\sigma}c_{k'\sigma}e^{irk'}\underbrace{\frac{1}{N}\sum_j e^{-i(k-k')j}}_{\delta_{kk'}} = \sum_k e^{irk}c^\dagger_{k\sigma}c_{k\sigma}. \tag{26}$$

Thus

$$H = \sum_{k\sigma}\left[-t\underbrace{(e^{ik}+e^{-ik})}_{2\cos k}+t'\underbrace{(e^{2ik}+e^{-2ik})}_{2\cos 2k}\right]c^\dagger_{k\sigma}c_{k\sigma} = \sum_{k\sigma}\varepsilon_k c^\dagger_{k\sigma}c_{k\sigma}. \tag{27}$$

(b) A sketch of ε_k for $t'=0$ is shown below. Each k-vector can accommodate 2 electrons (with opposite spin projections). Since the system is half-filled, the $N/2$ wavevectors with smallest energy ε_k will be occupied with 2 electrons each and the remaining $N/2$ wavevectors will be unoccupied. As ε_k has a minimum at $k=0$ and grows monotonically away from it (inside the 1st Brillouin zone), the occupied half of the wavevectors is the k-space region between $-\pi/2$ and $\pi/2$ (since the k-vectors are uniformly spaced). Therefore the Fermi wavevectors are at $k=\pm\pi/2$.



(c) When r is increased from 0, ε_k will for sufficiently large r develop a local maximum at $k = 0$ [1] that then grows with r. As long as the energy ε_0 of this local maximum is smaller than $\varepsilon_{\pm\pi/2}$, the occupied wavevectors will continue to lie in the interval $(-\pi/2, \pi/2)$ so that there continues to be 2 Fermi wavevectors, located at $k = \pm\pi/2$, just as for $r = 0$. However, at some critical value $r = r_c$ we get $\varepsilon_0 = \varepsilon_{\pm\pi/2}$. For $r > r_c$ a region of k-space around $k = 0$ will then become unoccupied, so that the occupied region of the 1st Brillouin zone splits into two symmetric parts, one on each side of $k = 0$. Thus for $r > r_c$ there will be 4 Fermi wavevectors. The critical value of r can be found from the condition

$$\varepsilon_0 = \varepsilon_{\pm\pi/2} \tag{28}$$

i.e.

$$-2t(\cos 0 - r\cos(2\cdot 0)) = -2t(\cos\pi/2 - r\cos(2\cdot\pi/2)) \Rightarrow 1 - r = -r(-1) \tag{29}$$

which gives $r = 1/2 = r_c$. The Fermi energy at $r = r_c$ is

$$\varepsilon_F = -2t(\cos 0 - \frac{1}{2}\cos(2\cdot 0)) = -t. \tag{30}$$

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[1] This local maximum arises when the second derivative of ε_k at $k = 0$ changes sign from positive to negative, which happens at $r = 1/4$:

$$\frac{d^2\varepsilon_k}{dk^2}|_{k=0} = 0 \Rightarrow [-\cos k + 4r\cos 2k]_{k=0} = 0 \Rightarrow -1 + 4r = 0 \Rightarrow r = 1/4.$$