TFY4210 Quantum theory of many-particle systems
Exam 2015
Solution

Problem 1

(a) $$b_\uparrow b_\uparrow^\dagger |n_\uparrow, n_\downarrow\rangle = \sqrt{n_\uparrow + 1}\, b_\uparrow |n_\uparrow + 1, n_\downarrow\rangle$$

$$= \sqrt{n_\uparrow + 1}\sqrt{n_\uparrow + 1}\,|(n_\uparrow + 1) - 1, n_\downarrow\rangle$$

$$= (n_\uparrow + 1)|n_\uparrow, n_\downarrow\rangle,$$

$$b_\uparrow^\dagger b_\uparrow |n_\uparrow, n_\downarrow\rangle = \sqrt{n_\uparrow}\, b_\uparrow^\dagger |n_\uparrow - 1, n_\downarrow\rangle$$

$$= \sqrt{n_\uparrow}\sqrt{(n_\uparrow - 1) + 1}\,|(n_\uparrow - 1) + 1, n_\downarrow\rangle$$

$$= n_\uparrow |n_\uparrow, n_\downarrow\rangle$$

$$\Rightarrow (b_\uparrow b_\uparrow^\dagger - b_\uparrow^\dagger b_\uparrow)|n_\uparrow, n_\downarrow\rangle$$

$$= (n_\uparrow + 1 - n_\uparrow)|n_\uparrow, n_\downarrow\rangle = |n_\uparrow, n_\downarrow\rangle$$

i.e. $$[b_\uparrow, b_\uparrow^\dagger]|n_\uparrow, n_\downarrow\rangle = 1\cdot|n_\uparrow, n_\downarrow\rangle.$$

As this result is valid for an arbitrary basis state $|n_\uparrow, n_\downarrow\rangle$, we obtain the operator identity

$$[b_\uparrow, b_\uparrow^\dagger] = 1 \qquad \text{QED.}$$

More generally, the commutation relations that follow from (1) can be written ($\alpha, \beta = \uparrow, \downarrow$)

$$[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}, \quad [b_\uparrow, b_\downarrow] = [b_\uparrow^\dagger, b_\downarrow^\dagger] = 0$$

(b) $\underline{\underline{[\hat{S}^+, \hat{S}^-]}} = b_\uparrow^\dagger b_\downarrow b_\downarrow^\dagger b_\uparrow - b_\downarrow^\dagger b_\uparrow b_\uparrow^\dagger b_\downarrow$

$= b_\uparrow^\dagger (1+\hat{n}_\downarrow) b_\uparrow - b_\downarrow^\dagger (1+\hat{n}_\uparrow) b_\downarrow$

$= \hat{n}_\uparrow (1+\hat{n}_\downarrow) - \hat{n}_\downarrow (1+\hat{n}_\uparrow)$

$= \hat{n}_\uparrow + \cancel{\hat{n}_\uparrow \hat{n}_\downarrow} - \hat{n}_\downarrow - \cancel{\hat{n}_\downarrow \hat{n}_\uparrow}$

$= \hat{n}_\uparrow - \hat{n}_\downarrow = \underline{\underline{2\hat{S}^z}}$,

(using bosonic comm. relations including that ↑- and ↓-operators commute)

$\underline{\underline{[\hat{S}^z, \hat{S}^+]}} = [\frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow), b_\uparrow^\dagger b_\downarrow]$

$= \frac{1}{2}[\hat{n}_\uparrow, b_\uparrow^\dagger b_\downarrow] - \frac{1}{2}[\hat{n}_\downarrow, b_\uparrow^\dagger b_\downarrow]$

$= \frac{1}{2}\underbrace{[\hat{n}_\uparrow, b_\uparrow^\dagger]}_{= b_\uparrow^\dagger} b_\downarrow - \frac{1}{2} b_\uparrow^\dagger \underbrace{[\hat{n}_\downarrow, b_\downarrow]}_{= -b_\downarrow}$

using Eqs. (23), (24)

$= b_\uparrow^\dagger b_\downarrow = \underline{\underline{\hat{S}^+}}$

$[\hat{S}^z, \hat{S}^-] = -\hat{S}^-$ can be proven in the same way. Alternatively, one can note that since the Schwinger boson expressions for $\hat{S}^+$ and $\hat{S}^-$ are h.c.'s of each other,

$\underline{\underline{[\hat{S}^z, \hat{S}^-]}} = \hat{S}^z\hat{S}^- - \hat{S}^-\hat{S}^z = (\hat{S}^+\hat{S}^z)^\dagger - (\hat{S}^z\hat{S}^-)^\dagger$

$= [\hat{S}^+, \hat{S}^z]^\dagger = (-\hat{S}^+)^\dagger = \underline{\underline{-\hat{S}^-}}$

(c) $\hat{\vec{S}}^2 = \hat{S}^x \hat{S}^x + \hat{S}^y \hat{S}^y + \hat{S}^z \hat{S}^z$

$$= \left(\frac{1}{2}\right)^2 (\hat{S}^+ + \hat{S}^-)(\hat{S}^+ + \hat{S}^-) + \left(\frac{1}{2i}\right)^2 (\hat{S}^+ - \hat{S}^-)(\hat{S}^+ - \hat{S}^-) + \hat{S}^z \hat{S}^z$$

$$= \frac{1}{2}\left(\hat{S}^+ \hat{S}^- + \hat{S}^- \hat{S}^+\right) + \hat{S}^z \hat{S}^z$$

$$= \frac{1}{2}\left(b_\uparrow^\dagger b_\downarrow b_\downarrow^\dagger b_\uparrow + b_\downarrow^\dagger b_\uparrow b_\uparrow^\dagger b_\downarrow\right) + \left(\frac{1}{2}\right)^2 (\hat{n}_\uparrow - \hat{n}_\downarrow)^2$$

$$= \frac{1}{2}\left[\hat{n}_\uparrow (1 + \hat{n}_\downarrow) + \hat{n}_\downarrow (1 + \hat{n}_\uparrow)\right] + \frac{1}{4}(\hat{n}_\uparrow - \hat{n}_\downarrow)^2$$

$$= \frac{1}{2}\left[\hat{n}_\uparrow + \hat{n}_\downarrow + 2\hat{n}_\uparrow \hat{n}_\downarrow\right] + \frac{1}{4}\left[\hat{n}_\uparrow^2 - 2\hat{n}_\uparrow \hat{n}_\downarrow + \hat{n}_\downarrow^2\right]$$

$$= \frac{1}{2}(\hat{n}_\uparrow + \hat{n}_\downarrow) + \frac{1}{4}(\hat{n}_\uparrow + \hat{n}_\downarrow)^2$$

$$= \underline{\underline{\frac{\hat{n}_\uparrow + \hat{n}_\downarrow}{2}\left[\frac{\hat{n}_\uparrow + \hat{n}_\downarrow}{2} + 1\right]}}$$

$$\Rightarrow \hat{\vec{S}}^2 \,|n_\uparrow, n_\downarrow\rangle$$

$$= \frac{\hat{n}_\uparrow + \hat{n}_\downarrow}{2}\left[\frac{\hat{n}_\uparrow + \hat{n}_\downarrow}{2} + 1\right]|n_\uparrow, n_\downarrow\rangle$$

$$= \frac{n_\uparrow + n_\downarrow}{2}\left[\frac{n_\uparrow + n_\downarrow}{2} + 1\right]|n_\uparrow, n_\downarrow\rangle$$

$$= \underline{\underline{S(S+1)\,|n_\uparrow, n_\downarrow\rangle}} \quad \text{if} \quad n_\uparrow + n_\downarrow = 2S$$

(d) $U(\theta) = e^{-i\hat{S}^z\theta/\hbar}$

Setting $\hbar = 1$ as usual and using the Schwinger boson representation gives

$$U(\theta) = \exp\left[-i\frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)\theta\right]$$

$$\Rightarrow \quad U(\theta)\, b_\alpha^\dagger\, U^\dagger(\theta) = e^{-i\frac{\theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)}\, b_\alpha^\dagger\, e^{i\frac{\theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)}$$

To calculate this we use the Baker-Hausdorff theorem (25):

$$e^{-B} A\, e^{B} = A + [A,B] + \frac{1}{2!}[[A,B],B] + \cdots$$

with $A = b_\alpha^\dagger$ and $B = i\frac{\theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)$

For $\alpha = \uparrow$, $[A,B] = \left[b_\uparrow^\dagger, i\frac{\theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)\right]$

$$= i\frac{\theta}{2}\left[b_\uparrow^\dagger, \hat{n}_\uparrow\right] = -i\frac{\theta}{2}\, b_\uparrow^\dagger\,,$$

where we used (24). As the result is just a constant $-i\frac{\theta}{2}$ times A, it follows that

$$[A,B]_n = \left(-i\frac{\theta}{2}\right)^n b_\uparrow^\dagger$$

Therefore

$$\underline{\underline{U(\theta)\, b_\uparrow^\dagger\, U^\dagger(\theta)}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\theta}{2}\right)^n b_\uparrow^\dagger = \underline{\underline{e^{-i\frac{\theta}{2}}\, b_\uparrow^\dagger}}$$

For $\alpha = \downarrow$, $[A,B] = i\frac{\theta}{2}\left[b_\downarrow^\dagger, -\hat{n}_\downarrow\right] = i\frac{\theta}{2}\, b_\downarrow^\dagger\,,$

thus giving $\underline{\underline{U(\theta)\, b_\downarrow^\dagger\, U^\dagger(\theta) = e^{i\frac{\theta}{2}}\, b_\downarrow^\dagger}}$

Use that $|S,m\rangle = |n_\uparrow, n_\downarrow\rangle$ with $n_{\uparrow\downarrow} = S \pm m$

and $|n_\uparrow, n_\downarrow\rangle \propto (b_\uparrow^+)^{n_\uparrow} (b_\downarrow^+)^{n_\downarrow} |0,0\rangle$

$$\Rightarrow U(2\pi)|S,m\rangle \propto U(2\pi) (b_\uparrow^+)^{S+m} (b_\downarrow^+)^{S-m} |0,0\rangle$$

$$= U(2\pi) \underbrace{b_\uparrow^+ b_\uparrow^+ \cdots b_\uparrow^+}_{S+m \text{ factors}} \underbrace{b_\downarrow^+ b_\downarrow^+ \cdots b_\downarrow^+}_{S-m \text{ factors}} |0,0\rangle$$

Now insert $I = U^+(2\pi) U(2\pi)$ between adjacent b_α^+ operators and after the last $b_\downarrow^+$

$$\Rightarrow \left(U(2\pi) b_\uparrow^+ U^+(2\pi)\right)^{S+m} \left(U(2\pi) b_\downarrow^+ U^+(2\pi)\right)^{S-m} U(2\pi)|0,0\rangle$$

Use $U(2\pi) b_\uparrow^+ U^+(2\pi) = e^{-i\frac{2\pi}{2}} b_\uparrow^+ = e^{-i\pi} b_\uparrow^+ = -b_\uparrow^+$

$U(2\pi) b_\downarrow^+ U^+(2\pi) = e^{i\frac{2\pi}{2}} b_\downarrow^+ = e^{i\pi} b_\downarrow^+ = -b_\downarrow^+$

$$U(2\pi)|0,0\rangle = e^{-i\frac{2\pi}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)} = \sum_{m=0}^{\infty} \frac{(-i\pi)^m}{m!} (\hat{n}_\uparrow - \hat{n}_\downarrow)^m |0,0\rangle$$

$= |0,0\rangle$ since only the $m=0$ term gives a nonzero contribution (since $\hat{n}_\uparrow|0,0\rangle = \hat{n}_\downarrow|0,0\rangle = 0$)

$$\Rightarrow U(2\pi)|S,m\rangle \propto (-b_\uparrow^+)^{S+m} (-b_\downarrow^+)^{S-m} |0,0\rangle$$

$$= (-1)^{S+m+S-m} (b_\uparrow^+)^{S+m} (b_\downarrow^+)^{S-m} |0,0\rangle$$

$$= (-1)^{2S} (b_\uparrow^+)^{S+m} (b_\downarrow^+)^{S-m} |0,0\rangle$$

i.e. $U(2\pi)|S,m\rangle = (-1)^{2S} |S,m\rangle$

which is what we wanted to verify using the Schwinger boson representation.

Remarks:

- Depending on at what stage one inserts the value $\theta = 2\pi$, one may end up with the factor $(-1)^{2m}$ instead of $(-1)^{2S}$. But these have the same value, since if $2S$ is an even (odd) integer, then $2m$ is also always even (odd).

- The result $U(2\pi)\,|S,m\rangle = (-1)^{2S}\,|S,m\rangle$ is trivial to verify from the general relations:

$$U(2\pi)\,|S,m\rangle = \exp\left(-i\cdot 2\pi\,\hat{S}_z\right)|S,m\rangle$$

$$= \exp\left(-i\cdot 2\pi m\right)|S,m\rangle = (-1)^{2m}\,|S,m\rangle = (-1)^{2S}\,|S,m\rangle$$

Problem 2

(a) $$c_{k+2\pi m} = \frac{1}{\sqrt{N}} \sum_j e^{-i(k+2\pi m)j} c_j$$

$$= \frac{1}{\sqrt{N}} \sum_j e^{-ikj} c_j \underbrace{e^{-i2\pi mj}}_{\substack{=1 \text{ since} \\ \text{both } m \text{ and } j}} = c_k \qquad (*)$$

Periodic BC's gives $k = 2\pi m/N$ (m integer). Pick k's to lie in region of length 2π (cf. $(*)$); we choose the standard region $-\pi \leq k < \pi$ (1BZ)

Let us define $H \equiv H_t + H_\Delta$

$$H_t = -t \sum_j \left(c_j^\dagger c_{j+1} + h.c. \right)$$

$$= -t \sum_j \left(\frac{1}{\sqrt{N}}\right)^2 \sum_{k,k'} e^{-ik'j} e^{ik(j+1)} c_{k'}^\dagger c_k + h.c.$$

$$= -t \sum_{k,k'} c_{k'}^\dagger c_k e^{ik} \underbrace{\frac{1}{N} \sum_j e^{i(k-k')j}}_{=\delta_{kk'}} + h.c.$$

$\leftarrow$ from using (28) with $F(k-k') = k-k'$

$$= -t \sum_k \left(e^{ik} + e^{-ik}\right) c_k^\dagger c_k = \underline{\underline{\sum_k \varepsilon_k c_k^\dagger c_k}}$$

where $\underline{\underline{\varepsilon_k = -2t \cos k}}$

We have $\varepsilon_{k\pm\pi} = -2t \cos(k \pm \pi)$

$$= -2t \left[\cos k \underbrace{\cos\pi}_{=-1} \mp \sin k \underbrace{\sin\pi}_{=0} \right] = -\varepsilon_k$$

$$H_\Delta = \Delta \sum_j (-1)^j c_j^\dagger c_j \qquad (\text{use } -1 = e^{i\pi})$$

$$= \Delta \sum_j e^{i\pi j} \left(\frac{1}{\sqrt{N}}\right)^2 \sum_{kk'} e^{-ik'j} e^{ikj} c_{k'}^\dagger c_k$$

$$= \Delta \sum_{kk'} c_{k'}^\dagger c_k \frac{1}{N} \sum_j e^{ij(k-k'+\pi)}$$

To calculate the j-sum we use (28) with $F(k-k') = k - k' + \pi$. The sum is thus 0 unless $k - k' + \pi = 2\pi m$ (m integer)

$\Rightarrow k - k' = (2m-1)\pi$

Given that both k and k' are in 1BZ, solutions are only possible for $m = 0, 1$, i.e. for $k - k' = \pm\pi$. Thus we may write

$$H_\Delta = \Delta \sum_{kk'} c_{k'}^\dagger c_k \sum_{p=\pm 1} \delta_{k', k+p\pi}$$

For terms with $k \in [-\pi, 0)$ we only get a contribution from $k' = k + \pi$ ($p = +1$)

$\Rightarrow \Delta c_{k+\pi}^\dagger c_k$

For terms with $k \in [0, \pi)$ we only get a contribution from $k' = k - \pi$ ($p = -1$)

$$\Rightarrow \Delta c_{k-\pi}^\dagger c_k = \Delta c_{k+\pi}^\dagger c_k$$

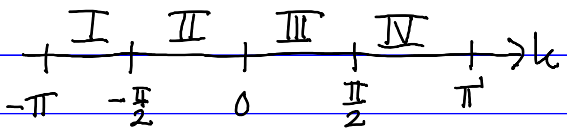
using $c_{k+\pi} = c_{k-\pi}$ which follows from $c_{k+2\pi} = c_k$

$$\Rightarrow \quad H_\Delta = \Delta \sum_k c_{k+\pi}^\dagger c_k$$

Thus we have shown that

$$\underline{\underline{H = \sum_{k \in 1BZ} \left[\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right]}}$$

(b) It is convenient to divide the 1BZ into 4 regions I, II, III, IV as shown below:



$$\Rightarrow \quad H = \left(\sum_{k\in \mathrm{I}} + \sum_{k \in \mathrm{II}} + \sum_{k\in \mathrm{III}} + \sum_{k \in \mathrm{IV}} \right) \left(\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right)$$

Regions II + III make up the MBZ, and we see that their contribution gives the first and fourth terms in (16). The contribution from region I is

$$\sum_{k \in \mathrm{I}} \left(\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right)$$

$$= \sum_{k\in \mathrm{III}} \Big[\underbrace{\varepsilon_{k-\pi}}_{-\varepsilon_k} \underbrace{c_{k-\pi}^\dagger}_{c_{k+\pi}^\dagger} \underbrace{c_{k-\pi}}_{c_{k+\pi}} + \Delta c_{k+\pi-\pi}^\dagger \underbrace{c_{k-\pi}}_{c_{k+\pi}} \Big]$$

$$= \sum_{k\in \mathrm{III}} \left[-\varepsilon_k c_{k+\pi}^\dagger c_{k+\pi} + \Delta c_k^\dagger c_{k+\pi} \right]$$

Similarly, the contribution from region IV is

$$\sum_{k \in \mathrm{IV}} \left(\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right)$$

$$= \sum_{k\in \mathrm{II}} \Big(\underbrace{\varepsilon_{k+\pi}}_{-\varepsilon_k} c_{k+\pi}^\dagger c_{k+\pi} + \Delta \underbrace{c_{k+\pi+\pi}^\dagger}_{c_k^\dagger} c_{k+\pi} \Big)$$

$$= \sum_{k\in \mathrm{II}} \left[-\varepsilon_k c_{k+\pi}^\dagger c_{k+\pi} + \Delta c_k^\dagger c_{k+\pi} \right]$$

Thus the contributions from regions I + IV give the second and third terms in (16). In conclusion,

$$H = \sum_{k \in \mathrm{MBZ}} \left[\varepsilon_k \left(c_k^\dagger c_k - c_{k+\pi}^\dagger c_{k+\pi} \right) + \Delta \left(c_k^\dagger c_{k+\pi} + c_{k+\pi}^\dagger c_k \right) \right]$$

(c) Each term in the k-sum can now be diagonalized separately. To save some writing we drop the k-index in the following and also define $c_k \equiv a$, $c_{k+\pi} \equiv b$. Thus we will diagonalize the operator

$$h \equiv \varepsilon\,(a^+ a - b^+ b) + \Delta\,(a^+ b + b^+ a)$$

$$\begin{aligned}
a^+ a &= (u\alpha^+ - v\beta^+)(u\alpha - v\beta) \\
&= u^2 \alpha^+\alpha + v^2 \beta^+\beta - uv(\alpha^+\beta + \beta^+\alpha) \\
b^+ b &= (v\alpha^+ + u\beta^+)(v\alpha + u\beta) \\
&= v^2 \alpha^+\alpha + u^2 \beta^+\beta + uv(\alpha^+\beta + \beta^+\alpha) \\
a^+ b &= (u\alpha^+ - v\beta^+)(v\alpha + u\beta) \\
&= uv(\alpha^+\alpha - \beta^+\beta) + u^2 \alpha^+\beta - v^2 \beta^+\alpha \\
b^+ a &= (a^+ b)^+ \\
&= uv(\alpha^+\alpha - \beta^+\beta) - v^2 \alpha^+\beta + u^2 \beta^+\alpha
\end{aligned}$$

$$\Rightarrow \quad h = F\,(\alpha^+\alpha - \beta^+\beta) + G\,(\alpha^+\beta + \beta^+\alpha)$$

where
$$\begin{aligned}
F &= \varepsilon(u^2 - v^2) + 2\Delta uv \\
G &= -2\varepsilon uv + \Delta(u^2 - v^2)
\end{aligned}$$

The off-diagonal terms (i.e. the terms prop. to $(\alpha^+\beta + \beta^+\alpha)$) vanish if $G = 0 \Rightarrow \Delta(u^2 - v^2) = \varepsilon \cdot 2uv$. Using the parametrization $u = \cos\theta$, $v = \sin\theta$, and (29)-(30), the condition $G = 0$ becomes $\Delta \cos 2\theta = \varepsilon \sin 2\theta$

$$\Rightarrow \quad \tan 2\theta = \frac{\Delta}{\varepsilon}$$

This gives furthermore

$$F = \varepsilon \cos 2\theta + \Delta \sin 2\theta = \cos 2\theta \left[\varepsilon + \Delta \tan 2\theta \right]$$

Using (31) gives $\cos 2\theta = \pm \dfrac{1}{\sqrt{1 + \tan^2 2\theta}} = \pm \dfrac{1}{\sqrt{1 + (\Delta/\varepsilon)^2}}$

Let us choose the positive sign (choosing the negative sign would however also be allowed and would not affect the physical results).

$$\Rightarrow \quad F = \frac{1}{\sqrt{1+\left(\frac{\Delta}{\varepsilon}\right)^2}}\left[\varepsilon + \Delta \frac{\Delta}{\varepsilon}\right] = \frac{\varepsilon^2+\Delta^2}{\varepsilon\sqrt{1+\left(\frac{\Delta}{\varepsilon}\right)^2}}$$

$$= -\sqrt{\varepsilon^2+\Delta^2}$$ (where we used that $\varepsilon = -|\varepsilon|$ for $k \in MBZ$)

This gives

$$H = \sum_{k\in MBZ}\left[E_k^{(\alpha)}\alpha_k^\dagger\alpha_k + E_k^{(\beta)}\beta_k^\dagger\beta_k\right]$$

where $$\underline{\underline{E_k^{(\beta)} = -E_k^{(\alpha)} = \sqrt{\varepsilon_k^2+\Delta^2}}}$$

Here $E_k^{(\alpha)}$ ($E_k^{(\beta)}$) is the energy of the α-mode $\alpha_k^\dagger|0\rangle$ (β-mode $\beta_k^\dagger|0\rangle$) with wavevector k (here $|0\rangle$ is the vacuum state with no fermions).

(d) As there are N wavevectors in 1BZ, there are $N/2$ wavevectors in MBZ. Thus there are $N/2$ α-modes and $N/2$ β-modes. Furthermore, for the generic case $t>0$ and $\Delta \neq 0$, $\sqrt{\varepsilon_k^2+\Delta^2}>0$ for all $k \in MBZ$. Thus the energy of any β-mode is higher than that of any α-mode. Invoking also the Pauli principle, it follows that in a system with $N/2$ fermions, the ground state $|G\rangle$ is characterized by each α-mode being occupied by one fermion while all β-modes are unoccupied. The ground state energy is therefore

$$E_G = \sum_{k\in MBZ} E_k^{(\alpha)} = -\sum_{k\in MBZ}\sqrt{\varepsilon_k^2+\Delta^2}$$

(The ground state $|G\rangle$ can be written $|G\rangle = \left(\prod_{k\in MBZ} \alpha_k^\dagger\right)|0\rangle$)

(e) $$\hat{N}_{even} - \hat{N}_{odd} = \sum_j (-1)^j c_j^\dagger c_j = \frac{1}{\Delta} H_\Delta$$

$$= \sum_{k\in MBZ} \left[c_k^\dagger c_{k+\pi} + c_{k+\pi}^\dagger c_k \right]$$

$$= \sum_{k\in MBZ} \left[2u_k v_k \left(\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k \right) + \left(u_k^2 - v_k^2\right)\left(\alpha_k^\dagger \beta_k + h.c.\right) \right]$$

Given the form of the ground state as discussed in (d), the ground state expectation values are

$$\langle \alpha_k^\dagger \alpha_k \rangle = 1, \quad \langle \beta_k^\dagger \beta_k \rangle = \langle \alpha_k^\dagger \beta_k \rangle = \langle \beta_k^\dagger \alpha_k \rangle = 0$$

$$\Rightarrow \langle (\hat{N}_{even} - \hat{N}_{odd}) \rangle = \sum_{k\in MBZ} 2u_k v_k = \sum_{k\in MBZ} \sin 2\theta_k$$

$$= \sum_{k\in MBZ} \cos 2\theta_k \tan 2\theta_k = \sum_{k\in MBZ} \frac{1}{\sqrt{1+(\Delta/\varepsilon_k)^2}} \frac{\Delta}{\varepsilon_k}$$

$$= \underline{\underline{- \sum_{k\in MBZ} \frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}}}}$$

(where we again used that $\varepsilon_k = -|\varepsilon_k|$ for $k \in MBZ$)

(i) $|\Delta| \ll t \Rightarrow |\langle(\hat{N}_{even} - \hat{N}_{odd})\rangle|$ small $\left(\propto \frac{|\Delta|}{t}\right)$

This is reasonable as in this case there is no appreciable energy difference between even and odd sites.

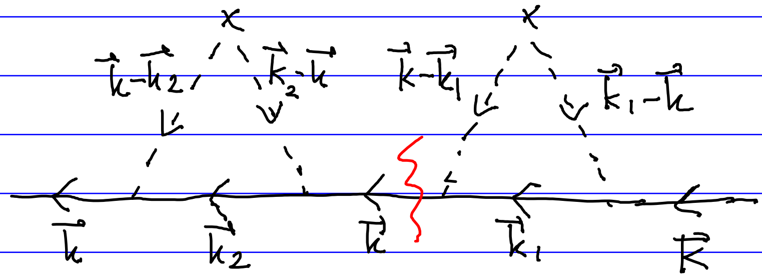
$|\Delta| \gg t \Rightarrow |\langle(\hat{N}_{even} - \hat{N}_{odd})\rangle| \approx \frac{N}{2}$

Reasonable; now strong preference for those $N/2$ sites with smallest energy cost, leaving the other $\frac{N}{2}$ sites $\approx$ empty

(ii) The sign equals $-\text{sign}(\Delta)$: reasonable, as it means odd sites preferred for $\Delta > 0$, even sites for $\Delta < 0$.

Problem 3

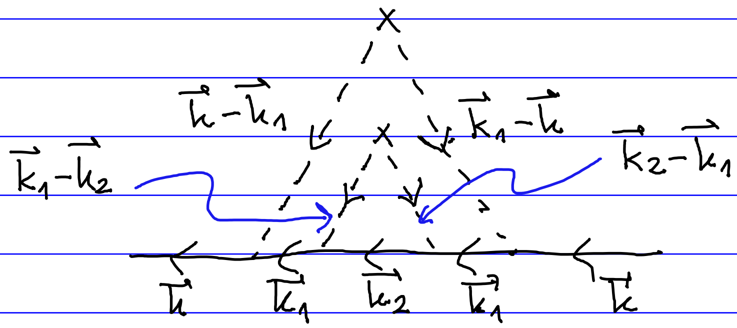
(a) Left diagram:



$$= N^2\left[g^{(0)}(\vec k)\right]^3 \sum_{\vec k_1 \vec k_2} U(\vec k_1-\vec k)\, g^{(0)}(\vec k_1)\, U(\vec k-\vec k_1)\, U(\vec k_2-\vec k)\, g^{(0)}(\vec k_2)\, U(\vec k-\vec k_2)$$

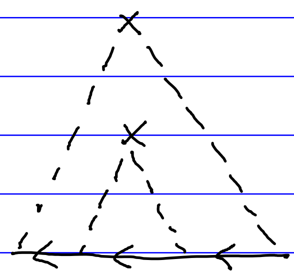
The diagram is reducible, as it falls apart by cutting the middle electron line (the cut is indicated by the red line)

Right diagram:



$$= N^2\left[g^{(0)}(\vec k)\right]^2 \sum_{\vec k_1 \vec k_2} U(\vec k_1-\vec k)\, g^{(0)}(\vec k_1)\, U(\vec k_2-\vec k_1)\, g^{(0)}(\vec k_2)\, U(\vec k_1-\vec k_2) \cdot g^{(0)}(\vec k_1)\, U(\vec k-\vec k_1)$$

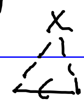
The diagram is irreducible, as it does not fall apart by cutting a single internal electron line. The corresponding self-energy diagram is

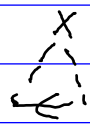


(b) 1. Any term in $\bar{G}$ can be written on the form

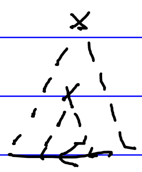
$$G^{(0)} \Sigma^{j_1} G^{(0)} \Sigma^{j_2} G^{(0)} \cdots G^{(0)} \Sigma^{j_m} G^{(0)}$$

where Σ^{j_i} is a self-energy diagram, and m is the number of self-energy diagrams in the term. The exact $\bar{G}$ is obtained by summing over m ($m = 0 \ldots \infty$) and for each m summing each factor Σ^{j_i} over all self-energy diagrams. Typically one however restricts to some approximative self-energy, corresponding to only summing each Σ^{j_i} over a certain subset of all self-energy diagrams. Even if that subset is finite, one still generates an infinite subset of Feynman diagrams from it, as there is no limit to the number m of self-energy diagrams in a term.

2. In the case of the 1st Born approximation, the only self-energy diagrams taken into account are  and 

The left Feynman diagram is included in this approximation, as it contains two self-energy diagrams of type  which is included in Σ_{1B}.

In contrast, the right Feynman diagram is not included, as it contains the self-energy diagram



which is not included in Σ_{1B}.

(c) The 1st Born approximation for $\bar{\mathcal{G}}$ becomes

$$\bar{\mathcal{G}}(\vec{k}, ip_m) = \frac{1}{ip_m - \xi_{\vec{k}} - \Delta + \frac{i}{2\tau}\,\mathrm{sgn}(p_m)}$$

$\overline{G^R}(\vec{k},\omega)$ is obtained from analytic continuation:

$$\overline{G^R}(\vec{k},\omega) = \bar{\mathcal{G}}(\vec{k}, ip_m)\Big|_{ip_m \to \omega + i\eta}$$

Thus the term ip_m in $\bar{\mathcal{G}}$ is replaced by $\omega + i\eta$, while $\mathrm{sgn}(p_m) = \mathrm{sgn}(\mathrm{Im}(ip_n))$ is replaced by $\mathrm{sgn}(\mathrm{Im}(\omega + i\eta)) \underset{\omega\ \text{real}}{=} \mathrm{sgn}(\mathrm{Im}(i\eta)) = \mathrm{sgn}(\eta) = +1$. This gives

$$\overline{G^R}(\vec{k},\omega) = \frac{1}{\omega + i\eta - \xi_{\vec{k}} - \Delta + \frac{i}{2\tau}} \overset{\tau\ \text{finite}}{\cong} \frac{1}{\omega - (\xi_{\vec{k}} + \Delta) + \frac{i}{2\tau}}$$

where in the last transition we could neglect the infinitesimal $i\eta$ since $\frac{i}{2\tau}$ is finite.

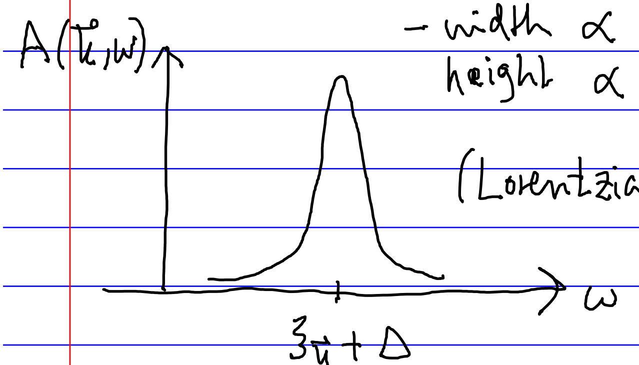
$$A(\vec{k},\omega) = -\frac{1}{\pi}\mathrm{Im}\,\overline{G^R}(\vec{k},\omega) = -\frac{1}{\pi}\mathrm{Im}\,\frac{\omega - (\xi_{\vec{k}}+\Delta) - \frac{i}{2\tau}}{(\omega - (\xi_{\vec{k}}+\Delta))^2 + \left(\frac{1}{2\tau}\right)^2}$$

$$= \underline{\underline{\frac{1}{\pi}\,\frac{1/2\tau}{(\omega - (\xi_{\vec{k}}+\Delta))^2 + (1/2\tau)^2}}}$$

(d)

- peak at $\xi_{\vec{k}} + \Delta$
- width $\propto 1/\tau$, height $\propto \tau$

(Lorentzian)



In the absence of scattering: $\Delta = 0$, $\tau = \infty$ $\left(\frac{1}{2\tau} \to \eta = 0^+\right)$

$\Rightarrow$ Dirac δ-function peaked at $\xi_{\vec{k}}$