Solution

Exam TFY4210

May 12, 2020

Note: Students were allowed to use lecture notes during the exam.

Problem 1

a) Using the operators and following the steps in lecture notes, we obtain

$$E_k = -\gamma_k - \mu$$

$$\gamma_k = \sum_{\vec{\delta}} t(\delta)\, e^{i\vec{k}\cdot\vec{\delta}}$$

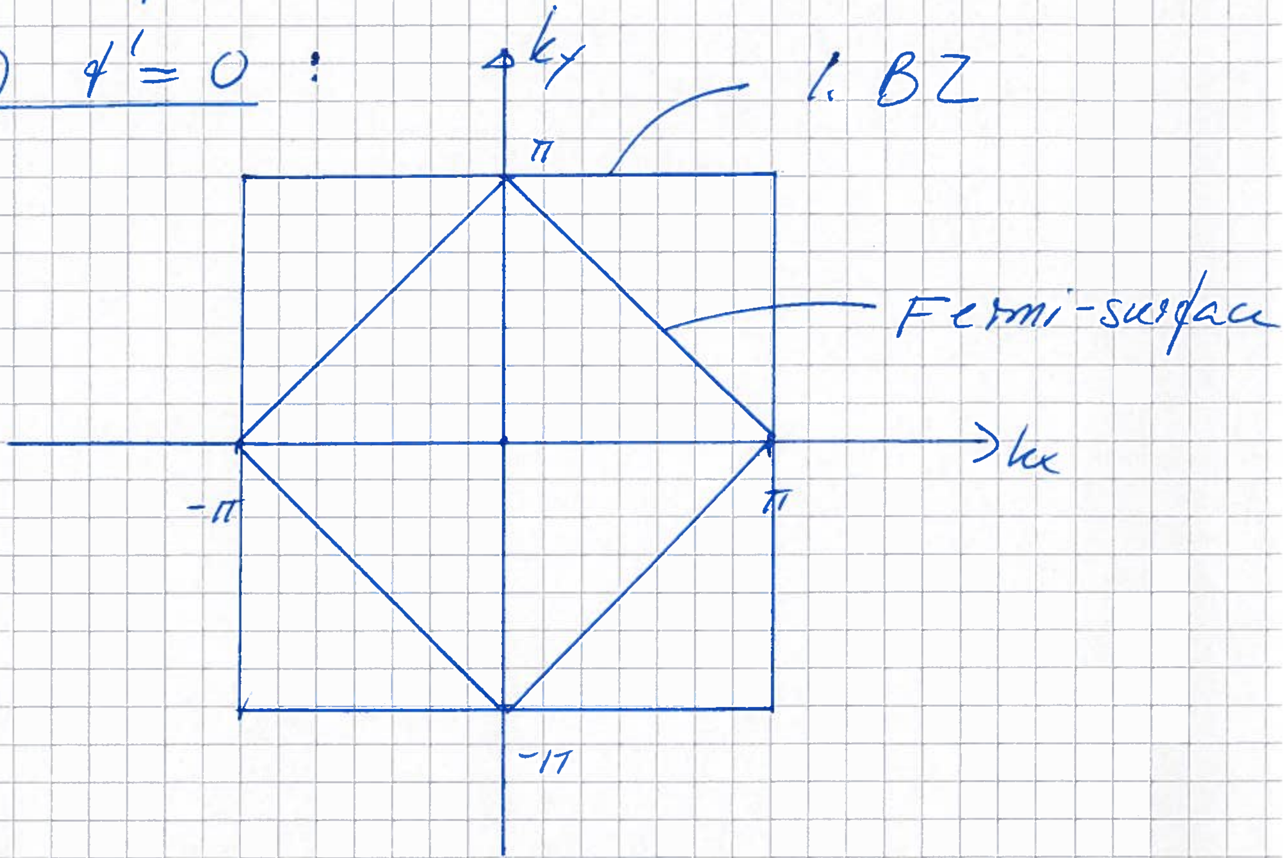
$$\vec{\delta} = \vec{r}_i - \vec{r}_j$$

b) For a 2D square lattice, with nearest-neighbor hopping t and next-nearest hopping t', we find

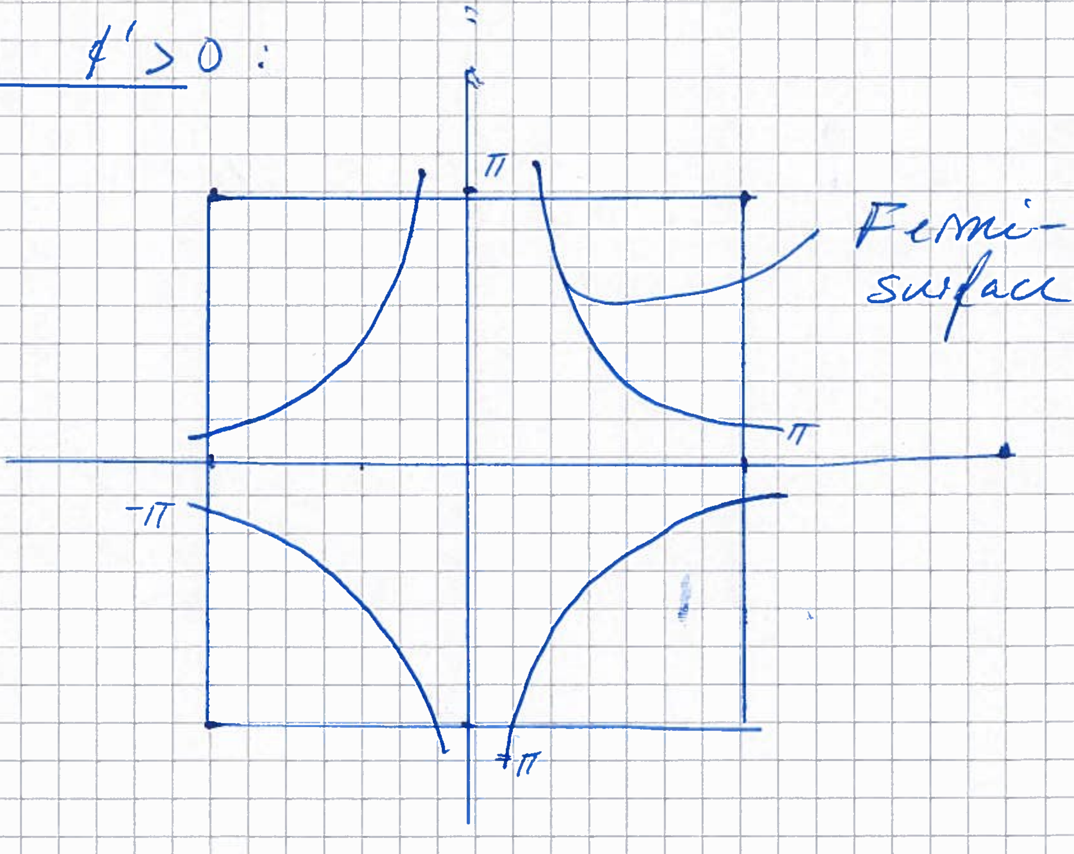
$$\gamma_k = -2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cdot \cos k_y$$

c) $\mu = 0$ Fermi-surface

i) $t' = 0$:



ii) $t' > 0$:



d)
$$\frac{\langle N_f \rangle}{N} = \frac{1}{N} \sum_{\vec{k},\sigma} \langle n_{k\sigma} \rangle$$

$$= \frac{1}{N} \sum_{\vec{k},\sigma} \frac{1}{e^{\beta E_k} + 1}$$

$$= \frac{2}{N} \sum_{\vec{k}} \frac{1}{e^{\beta E_k} + 1}$$

$$E_k = -\gamma_k - \mu$$

Oppgave 2

The model is the same as treated in lectures, with an additional anistropy term

$$-K \sum_i S_{iz}^2$$

$J < 0 \; ; \; K > 0$

The spectrum has been found by detailed calculations with $K=0$ in class. In that case, we found

$$H = \tilde{E}_0 + \sum_q \omega_q \left(A_q^+ A_q + \frac{1}{2} + B_q^+ B_q + \frac{1}{2} \right)$$

$$\omega_q = 2 |J| S \sqrt{z^2 - \gamma_q^2}$$

$z = \#$ number of nearest neighbors

$$\gamma(q) = \sum_{\vec{\delta}} e^{i \vec{q} \cdot \vec{\delta}}$$

$$\tilde{E}_0 = 2 N J z S(S+1)$$

$N = \#$ lattice sites on each sub-lattice

a) Here, the candidates are supposed to realize that by following the procedure for solving this problem used in class (for $K=0$), one expands to second order in boson-operators via the Holstein-Primakoff transformation.

At one intermediate stage, one has

$$H = E_0 - 2JSz \sum_{\vec{q}} \left(a_q^+ a_q + b_q^+ b_q \right)$$
$$- 2JS \sum_{\vec{q}} \gamma(\vec{q}) \left(a_q b_q + b_q^+ a_q^+ \right)$$

By using the same transformation on the anisotropy-term keeping terms to quadratic, this modifies $\tilde{E}_0$, and also leads to an additional term like the second term above. One may view this as a shift of the parameter z.

The additional term like the second, is

$$2KS\sum_{\vec{q}}\left(a_q^+ a_q + b_q^+ b_q\right)$$

which we may add to the second term, giving

$$-\left(2JSz - 2KS\right)\sum_{\vec{q}}\left(a_q^+ a_q + b_q^+ b_q\right)$$

$$= -2JSz'\sum_{\vec{q}}\left(a_q^+ a_q + b_q^+ b_q\right),$$

$$\underline{z' = z + \frac{K}{|J|}}$$

The Hamiltonian is now on precisely the same form as for $K=0$, and we obtain ω_q by copying the result from lecture notes

$$\underline{\omega_q = 2|J|S\sqrt{\left(z + \frac{K}{|J|}\right)^2 - \gamma_q^2}}\,,\quad q \in \text{reduced Brillouin-zone}$$

__b)__ For $K=0$ this has been done in lecture notes.

Considering magnetization along z-direction on one sublattice, we have

$$M = \frac{1}{N}\left\langle \sum_{i\in A} S_{iz} \right\rangle_A$$

where A denotes one sublattice and where M is the magnetization per lattice site

$$M = S - \frac{1}{N}\sum_q \langle a_q^+ a_q \rangle$$

where q runs over a reduced Brillouin-zone. The above average is taken with respect to the diagonalized Hamiltonian.

$$a_q = u_q A_q - v_q B_q^+$$
$$a_q^+ = u_q A_q^+ - v_q B_q$$

We thus find

$$\langle a_q^+ a_q \rangle = u_q^2 \langle A_q^+ A_q \rangle + v_q^2 \langle B_q B_q^+ \rangle$$
$$= u_q^2 \langle A_q^+ A_q \rangle + v_q^2 \left[\langle B_q^+ B_q \rangle + 1 \right]$$

$$M = S - \frac{1}{N}\sum_q v_q^2 - \frac{1}{N}\sum_q \left(\frac{u_q^2 + v_q^2}{e^{\beta\omega_q} - 1} \right)$$

Low-temp corrections:

The term involving the Bose-distribution is exponentially suppressed as $T \to 0$, so this term is negligible. Thus, we have

$$M = S - \frac{1}{N} \sum_q v_q^2$$

Only quantum-fluctuations are left.

c)

From lecture notes:

$$u_q^2 - v_q^2 = 1$$

$$u_q^2 = \frac{1}{2}\left(1 + \frac{z'}{\sqrt{z'^2 - \gamma_q^2}}\right)$$

$$v_q^2 = \frac{1}{2}\left(-1 + \frac{z'}{\sqrt{z'^2 - \gamma_q^2}}\right)$$

$$z' = z + \frac{K}{|J|}$$

$$\lim_{q \to 0} \gamma_q = z$$

$$\omega_q = 2|J|S\sqrt{z'^2 - \gamma_q^2}$$

ω_q: Finite when $q \to 0$, $K \neq 0$!

Thermal fluctuations are completely suppressed at low T due to the gap in ω_q as $q \to 0$.

Quantum-fluctuations are also small, since v_q^2 is also finite when $q \to 0$.