Answers to Exam TFY4225 H09 (30/11/2009)

Problem 1.

a) The atomic mass of nuclide (A, Z):

$$m(A,Z) = Z m(^1H) + N \cdot m_n - B/c^2, \qquad A = Z + N.$$

where $m(^1H)$ is the atomic mass of hydrogen, m_n is the mass of the neutron.

Binding energy:

$$B = a_1 A - a_2 \cdot A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(N-Z)^2}{A} \pm \Delta$$

$a_1 \cdot A$: Volume term expressing the constant contribution to the binding energy per nucleon due to strong interaction with nearest neighbours through short range nuclear force.

$-a_2 A^{2/3}$: Decrease in binding energy due to surface effect, i.e. lack of binding partners outside surface of nucleus. Analogous to surface energy of liquid droplet.

$-a_3 \frac{Z^2}{A^{1/3}}$: Decrease in binding energy due to Coulomb repulsion between protons.

$-a_4 \frac{(N-Z)^2}{A}$: Decrease in binding energy if number of neutrons differ from number of protons.

This is due to differences in both kinetic and potential energy if $N \neq Z$.

Δ: Pairing contribution which gives increased binding for even-even nuclei (+ sign), decreased binding energy for odd-odd nuclei (− sign.)
The Δ term is zero for A odd nuclei (i.e. odd-even or even-odd nuclei)

b) The most stable nuclide is the one with the smallest mass. This is determined as the value of Z corresponding to the minimal value of $m(A,Z)$ for fixed A.

For fixed A:

$$m(Z) = Zm(^1H) + (A-Z)m_n - \frac{1}{c^2}\left[a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(A-2Z)^2}{A} \pm \Delta\right]$$

$$\frac{\partial m}{\partial Z} = m(^1H) - m_n + \frac{1}{c^2}\left[\frac{2Za_3}{A^{1/3}} + \frac{2a_4(A-2Z)(-2)}{A}\right]$$

$$\frac{\partial m}{\partial Z} = 0 \Rightarrow (m(^1H) - m_n)c^2 - 4a_4\frac{A}{A} + Z\left(\frac{2a_3}{A^{1/3}} + \frac{8a_4}{A}\right) = 0$$

$$\Rightarrow Z_{min} = \frac{[m_n - m(^1H)]c^2 + 4a_4}{2\frac{a_3}{A^{1/3}} + 8\frac{a_4}{A}}$$

$$\frac{\partial^2 m}{\partial Z^2} = \frac{2a_3}{A^{1/3}} + \frac{8a_4}{A} > 0$$ i.e. The value of $m(Z_{min})$ is indeed a minimum value.

Numerical values:

$$Z_{min} = \frac{(1{,}008665 - 1{,}007825)931{,}5 + 4\cdot 23{,}28}{2\cdot\frac{0{,}58}{4{,}626} + \frac{8\cdot 23{,}28}{99}} = 44$$ q.e.d.

Problem 2.

a) Filling of the nuclear shell model

	max no of nucleons in each shell	accumulated max number
$1g\,7/2$	8	64
$2d\,5/2$	6	56
$1g\,9/2$	10	50
$2p\,1/2$	2	40
$1f\,5/2$	6	38
$2p\,3/2$	4	32
$1f\,7/2$	8	28
$1d\,3/2$	4	20
$2s\,1/2$	2	16
$1d\,5/2$	6	14
$1p\,1/2$	2	8
$1p\,3/2$	4	6
$1s\,1/2$	2	2

Configuration for ${}^{99}_{43}Tc_{56}$:

$43P: \; (1s\,1/2)^2 \cdots (1g\,9/2)^3$

$56N: \; (1s\,1/2)^2 \cdots (2d\,5/2)^6$

The 56 neutrons completely fill shells up to and including $2d\,5/2$. Complete spin pairing and no net contribution to nuclear spin. The nuclear spin is determined by the odd proton in $1g\,9/2$, i.e. $I = 9/2$, with parity $(-1)^{\ell} = (-1)^4 = +1$, thus $I^{\pi} = \frac{9}{2}^{+}$ for ${}^{99}Tc$, q.e.d.

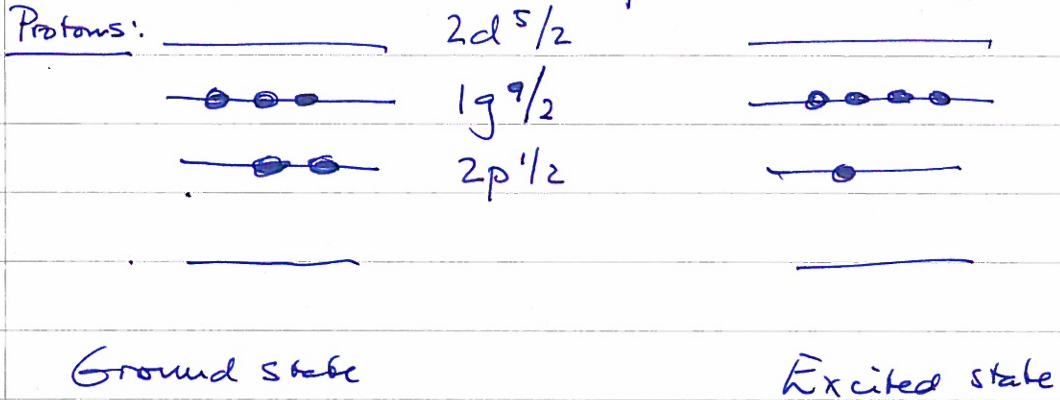
${}^{99}_{44}Ru_{55}$: $44P: (1s^1/2)^2 \ldots (1g^9/2)^4$

$55N: (1s^1/2)^2 \ldots (2d^5/2)^5$

Nuclear spin is determined by the odd neutron in $2d^5/2$, i.e. $I = 5/2$ with parity $(-1)^\ell = (-1)^2 = +1$.

Thus $I^\pi = 5/2^+$ for ${}^{99}Ru$, q.e.d.

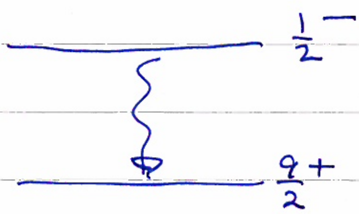
The excited state $1/2^-$ for ${}^{99m}Tc$ is formed as follows:



A proton is raised from the $2p^1/2$ state to the $1g^9/2$ state, thus giving an even number of protons in $1g^9/2$ with no spin contribution (due to pairing). Spin and parity will be determined by the odd proton in $2p^1/2$, i.e. $I = 1/2$, parity $(-1)^\ell = (-1)^1 = -1$, i.e. the state $1/2^-$

q.e.d.

b) Multipole contributions in the γ-emission from ${}^{99m}Tc$:



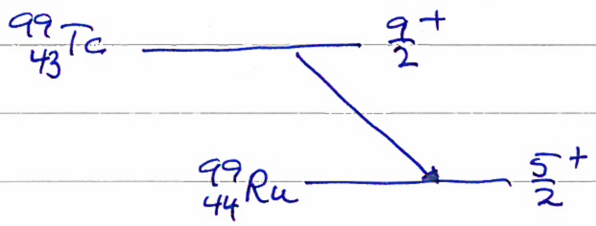
Parity shift: Possible multipoles: E1, M2, E3, M4, E5, ...

$$|I_i - I_f| \leq L \leq |I_i + I_f|$$
$$\left|\tfrac{1}{2} - \tfrac{9}{2}\right| \leq L \leq \left|\tfrac{1}{2} + \tfrac{9}{2}\right|$$
$$\underline{4 \leq L \leq 5}$$

The lowest possible multipole, M4, will dominate the transition, with possible contribution from E5.

The high multipolarity M4 will correspond to an unusually long half life. The indicated half life of 6 hrs is very long for γ-transitions (although not for α or β transitions). This is in agreement with ^{99m}Tc being labeled as a metastable state.

c) Classification of the β-transition $\frac{9}{2}^+ \rightarrow \frac{5}{2}^+$

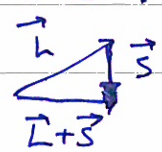


No parity shift : $\Rightarrow$ $L = 0, 2, 4 \ldots$

$$|I_i - I_f| \leq |\vec{L} + \vec{S}| \leq |I_i + I_f|$$
$$\left|\tfrac{9}{2} - \tfrac{5}{2}\right| \leq |\vec{L} + \vec{S}| \leq \left|\tfrac{9}{2} + \tfrac{5}{2}\right|$$
$$\underline{2 \leq |\vec{L} + \vec{S}| \leq 7}$$

The lowest possible value of L will be 2, in combination with S=0 or S=1.

For S=0 $\vec{L}+\vec{S} = \vec{L} = \vec{2}$. For S=1 :



$|\vec{L}+\vec{S}| = |\vec{L}| = 2.$

Thus, the transition is a second forbidden, mixed Fermi and Gamow Teller

The fermi integral has the value $f = 0.2$.

Thus $\log_{10}(f t_{1/2}) = \log_{10}(0,2) + \log_{10}\left(2.111 \cdot 10^5\,\mathrm{yr} \cdot 365\,\frac{\mathrm{d}}{\mathrm{yr}} \cdot 24\,\frac{\mathrm{hr}}{\mathrm{d}} \cdot 3600\,\frac{\mathrm{s}}{\mathrm{hr}}\right)$

$$\log_{10}(f t_{1/2}) = \log_{10}(0,2) + \log_{10}(6,657 \cdot 10^{12}) = -0,7 + 12,82 \simeq 12$$

$\log(f t_{1/2}) = 12$ is in good agreement with a 2nd forbidden transition

Problem 3.

a) Mo-column holding 1 MBq of ^{99}Mo on delivery at $t = 0$. Also, $N(^{99m}Tc, t=0) = 0$

Differential equations:

^{99}Mo: $\frac{dN_{Mo}}{dt} = -\lambda_{Mo} N_{Mo}$

$$\Rightarrow \quad N_{Mo}(t) = N_{Mo}(0) \cdot e^{-\lambda_{Mo} t} \qquad N_{Mo}(0) = N_0$$

^{99m}Tc: $\frac{dN_{Tc}(t)}{dt} = +\lambda_{Mo} N_{Mo}(t) - \lambda_{Tc} N_{Tc}(t)$

Trial solution: $N_{Tc}(t) = a e^{-\lambda_{Mo} t} + b e^{-\lambda_{Tc} t}$

Inserting trial solution:

$$-\lambda_{Mo}\cdot a e^{-\lambda_{Mo}t} - \lambda_{Tc}\cdot b e^{-\lambda_{Tc}t} = \lambda_{Mo} N_0 e^{-\lambda_{Mo}t} - \lambda_{Tc} a e^{-\lambda_{Mo}t} - \lambda_{Tc} b e^{-\lambda_{Tc}t}$$

Coeff. of terms including $e^{-\lambda_{Mo}t}$: $-\lambda_{Mo}\cdot a = \lambda_{Mo}\cdot N_0 - \lambda_{Tc}\cdot a$

$$\Rightarrow a = \frac{\lambda_{Mo}\cdot N_0}{\lambda_{Tc}-\lambda_{Mo}}$$

Coeff. of terms including $e^{-\lambda_{Tc}t}$: $-\lambda_{Tc}\cdot b = -\lambda_{Tc}\cdot b$ i.e. OK.

From the initial condition: $N_{Tc}(0)=0 \Rightarrow b=-a$.

$$\Rightarrow N_{Tc}(t) = \frac{\lambda_{Mo}}{\lambda_{Tc}-\lambda_{Mo}}\cdot N_0\left(e^{-\lambda_{Mo}t} - e^{-\lambda_{Tc}t}\right)$$

At $t=0$: $A_{Mo} = \lambda_{Mo}\cdot N_{Mo} = 1\ \text{MBq} = \lambda_{Mo}\cdot N_0 = A_0$

^{99m}Tc activity at 24hrs:

$$A_{Tc}(24\,\text{hrs}) = \lambda_{Tc}\cdot N_{Tc}(24\,\text{hrs}) = \frac{\lambda_{Tc}}{\lambda_{Tc}-\lambda_{Mo}}\cdot A_0\left[e^{-\lambda_{Mo}t} - e^{-\lambda_{Tc}t}\right]$$

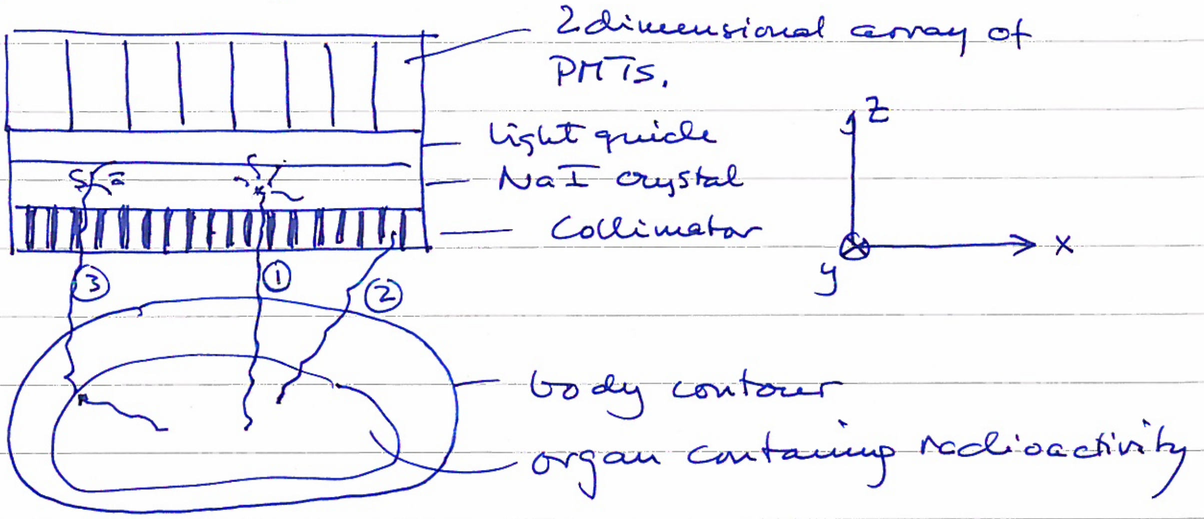
$$\lambda_{Mo} = \frac{\ln 2}{T_{1/2\,Mo}} = \frac{0{,}693}{65{,}94\,h} = 0{,}0105\,h^{-1}$$

$$\lambda_{Tc} = \frac{\ln 2}{T_{1/2\,Tc}} = \frac{0{,}693}{6{,}01\,h} = 0{,}1153\,h^{-1}$$

$$A_{Tc}(24\,\text{hrs}) = \frac{0.1153}{0{,}1153-0{,}0105}\cdot 1\,\text{MBq}\left(e^{-0{,}0105\cdot 24} - e^{-0.1153\cdot 24}\right)$$

$$A_{Tc}(24\,\text{hrs}) = 1.1\cdot(0{,}772-0{,}0628)\,\text{MBq} = 0{,}786\,\text{MBq} = 786\,\text{kBq}.$$

b) The gamma camera is designed to image the distribution of a radiopharmaceutical within the patient's body.



Each PMT is assigned an address X_i, Y_j in the two dimensional image plane xy.
The signal from PMT (i,j) is designated S_{ij}.

Only γ-emissions similar to the one labeled 1 will represent a true image by projection of the activity in the organ onto the image plane.
Emissions with oblique incidence (e.g. 2 in figure) will be stopped by the collimator.
Compton scattered photons (3 in figure) will be discriminated out based on an energy signal E:

$$E = \sum_{ij} S_{ij}, \quad \text{by requiring } E \in (\text{full energy peak})$$

for registration of the emission.

Location of the count in the image plane will be based on X and y addresses generated as:

$$x = \frac{\sum_{ij} x_i \cdot s_{ij}}{\sum_{ij} s_{ij}} \qquad y = \frac{\sum_{ij} y_i \, s_{ij}}{\sum_{ij} s_{ij}}$$

Each detection event is occurring as in a conventional NaI - PMT detector.

The result is an acquired image where each (x,y)-pixel value corresponds to a line integral of the activity in the object, along a line normal to the NaI crystal (i.e. image plane) in the position (x,y)

(These acquired data may be used for extracting projection profiles for use in computed tomography (SPECT) if many different angles of acquisition are used.)

c) Worst case estimate of committed effective dose after injection of 0.2 MBq ^{99m}Tc:

The nuclide is distributed in the whole body ($w_T = 1{,}0$), with no physiological excretion ($\lambda_B = 0$).

Equivalent dose to organ T: $H_T = \sum_S \tilde{A} \cdot SEE(T \leftarrow S)$

Specific effective energy: $SEE = \frac{1}{M_T} \sum w_{Ri} \, k_i \, \overline{E_i} \, \varphi_i$

$w_{Ri} = 1$ for β and γ.

$k_i = 1{,}0$ for γ-emission from ^{99m}Tc and for β-emission from ^{99}Tc.

First, consider γ emission from ^{99m}Tc:

$$SEE_{Tc} = \frac{1}{M_T} \cdot \sum w_{Ri} \cdot k_i \cdot \overline{E}_i \, \varphi_i$$

$$SEE_{Tc} = \frac{1 \frac{Sv}{Gy}}{70 kg} \cdot 142 \, keV \cdot 0{,}36 \cdot 1{,}602 \cdot 10^{-16} \frac{J}{keV}$$

$$\underline{\underline{SEE_{Tc} = 1{,}17 \cdot 10^{-16} \, Sv/dis.}}$$

Total number of disintegrations:

$$\tilde{A} = A_0 \int_0^{50 yrs} e^{-\lambda_{eff} t} \, dt$$

$T_{1/2 \, eff}$ for ^{99m}Tc is determined by the radiological half life $T_{1/2} = 6{,}01$ hrs, since $T_{1/2 \, biol} = \infty$

$$\left(T_{1/2 eff} = \frac{T_{1/2 R} \cdot T_{1/2 B}}{T_{1/2 R} + T_{1/2 B}} \longrightarrow T_{1/2 R} \text{ for } T_{1/2 B} \to \infty \right)$$

Since $T_{1/2 \, eff} = T_{1/2 R} = 6{,}01$ hr $\ll$ 50 yrs, the integral for $\tilde{A}$ is:

$$\tilde{A} \simeq A_0 \int_0^{\infty} e^{-\lambda_{eff} \cdot t} \, dt = \frac{A_0}{\lambda_{Tc, eff}} \qquad \lambda_{eff} = \lambda_{Tc} = 0{,}1153 \, h^{-1}$$

$$\underline{\underline{\tilde{A}}} = \frac{0{,}2 \, MBq \cdot 10^6 \frac{Bq}{MBq}}{0.1153 \, hr} \cdot 3600 \frac{s}{hr} = \underline{\underline{6{,}244 \cdot 10^9 \, Bq \cdot s = 6{,}244 \cdot 10^9 \, dis.}}$$

Committed effective dose from ^{99m}Tc γ-emission:

$$E_\gamma(50) = \sum w_T H_T(50) \qquad T \text{ corresponds to the whole body, } w_T = 1{,}0$$

$$\underline{\underline{E_\gamma(50)}} = \tilde{A} \cdot SEE = 1{,}17 \cdot 10^{-16} \frac{Sv}{dis} \cdot 6{,}244 \cdot 10^9 \, dis = \underline{\underline{7{,}3 \cdot 10^{-7} \, Sv = 0{,}73 \, \mu Sv}}$$

The "daughter" nuclide of ^{99m}Tc, ^{99}Tc, is also radioactive, having a β-disintegration with $T_{1/2} = 2.11 \cdot 10^5$ yrs.
Due to the very long half-life, the β^- dose from ^{99}Tc will be negligible over a 50 yrs time span.

Estimate of the contribution from $^{99}Tc \xrightarrow{\beta^-} {}^{99}Ru$:

$$SEE_\beta = \frac{1}{M_T} \cdot \sum \underbrace{w_R}_{=1} \underbrace{k_i}_{=1} \underbrace{\overline{E_i}}_{\simeq \frac{1}{3} E_{max} = \frac{1}{3} \cdot 293.7 keV} \underbrace{\varphi_i}_{=1 \text{ for } \beta}$$

$$SEE_\beta = \frac{1 \frac{Sv}{Gy}}{70 kg} \cdot \frac{293.7 \text{ keV}}{3 \text{ dis}} \cdot 1.602 \cdot 10^{-16} \frac{J}{keV} \cdot 1 = 2.24 \cdot 10^{-16} \text{ Sv/dis}$$

All ^{99m}Tc nuclei will deexcite into ^{99}Tc during a relatively short periode (days).
The number of ^{99}Tc nuclei that will give rise to β-activity:

$$N_{^{99}Tc} = N_{^{99m}Tc}(t=0) = \frac{A_{^{99m}Tc}(t=0)}{\lambda_{^{99m}Tc}}$$

$$N_{^{99}Tc} = \frac{0.2 \text{ MBq}}{0.1153 h^{-1} \cdot \frac{1}{3600} \frac{h}{s}} = 6244 \cdot 10^6 = 6.244 \cdot 10^9 \; (= \tilde{A}_{Tc})$$

Number of ^{99}Tc disintegrations over 50 yrs:

$$\tilde{A} = \int_0^T A_0 e^{-\lambda t} dt = \frac{A_0}{\lambda}\left[1 - e^{-\lambda T}\right] \simeq A_0 T \quad \text{for } \lambda T \ll 1.$$

$$\Rightarrow \tilde{A} = \lambda_{Tc\text{-}99} \cdot N_{Tc\text{-}99} \cdot T$$

$$\lambda_{Tc-99} = \frac{\ln 2}{T_{1/2\,Tc-99}} = \frac{0{,}693}{2{,}111 \cdot 10^5 \text{ yr}} = \underline{0{,}326 \cdot 10^{-5} \text{ yr}^{-1}}$$

$$\lambda_{Tc-99} \cdot T = 0{,}328 \cdot 10^{-5} \text{ yr}^{-1} \cdot 50 \text{ yr} = \underline{16{,}5 \cdot 10^{-5}}$$

$$\underline{\tilde{A}} = \lambda_{Tc-99} \cdot N_{Tc-99} \cdot T = 6{,}24 \cdot 10^{9} \cdot 16{,}4 \cdot 10^{-5} = \underline{1{,}023 \cdot 10^{6} \text{ dis}}$$

$$E_{\beta^-}(50) = \underbrace{w_T}_{=1{,}0} H_T(50) = \tilde{A} \cdot SEE = 1{,}023 \cdot 10^{6} \cdot 2{,}24 \cdot 10^{-16} \text{ Sv}$$

$E_{\beta^-}(50) = 2{,}3 \cdot 10^{-10}$ Sv, which is negligible compared to $E_\gamma(50)$

(The above estimation of $E_\beta(50)$ is not considered necessary for a complete answer, but the ~~....~~ statement on top of page 11, concluding that the contribution to $E(50)$ from β^- decay of Tc-99, is necessary.)

Total committed ~~affective~~ effective dose after injection of 0.2 MBq ^{99m}Tc is therefore $\underline{\underline{E(50) = E_\gamma(50) = 0{,}73\,\mu\text{Sv}}}$

Dose limit for the public is 1 mSv/yr.
Committed effective dose refers to the year of intake, but in this case $E(50) \ll 1$ mSv, thus of no consequence.

The probability for cancer development is 5%/Sv, thus the present $E(50)$ is of no consequence ($\sim 10^{-6}$)

Problem 4.

$$\mu = \underset{\text{photo electric}}{\tau} + \underset{\text{Compton}}{\sigma} + \underset{\text{Pair production}}{\kappa}$$

a) Linear attenuation coefficient μ expresses the relative fraction of incoming photon energy fluence that is removed from the collimated primary beam per unit length:

$$\mu = -\frac{1}{\Psi} \cdot \frac{d\Psi}{dx} \Rightarrow \Psi(x) = \Psi(0)\,e^{-\mu x}$$

The energy transfer coefficient μ_{tr} expresses the relative fraction of the incoming photon energy fluence that is converted into kinetic energy of secondary electrons due to ionization of the medium.

$$\underline{\mu_{tr} = \tau\left(1 - \frac{\delta}{h\nu}\right) + \sigma\left(1 - \frac{h\nu'}{h\nu}\right) + \kappa\left(1 - \frac{2m_ec^2}{h\nu}\right)}$$

for each term, the part which remains as photon radiation is subtracted:
For the photo electric effect, δ is the average energy of X-ray emission after an photo electric interaction
For Compton, $h\nu'$ is the energy of the compton photon.
For pair production, $2m_ec^2$ is the total energy of the two annihilation photons that will be emitted after the positron has lost its kinetic energy and interacts with an electron of the medium.

The energy absorption coefficient corrects μ_{tr} for ~~energy~~ bremsstrahlung emission by the secondary electrons, thus $\underline{\mu_{en} = \mu_{tr}(1-g)}$

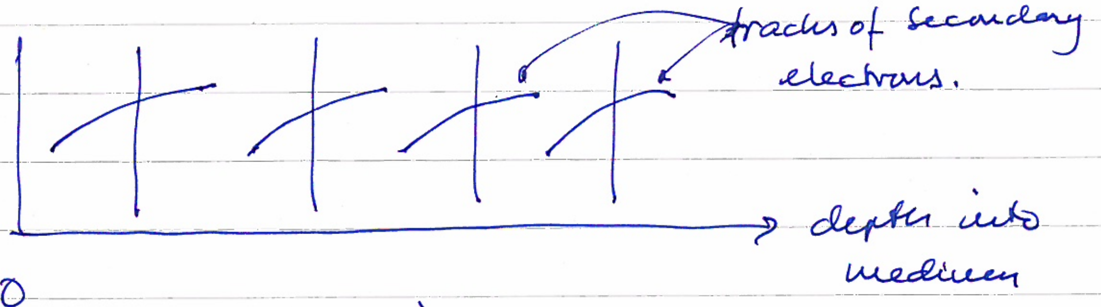
Absorbed dose : $$D = \lim_{\Delta V \to 0} \frac{\bar{\varepsilon}}{\rho \Delta V}$$

Kerma (kinetic energy released per mass):

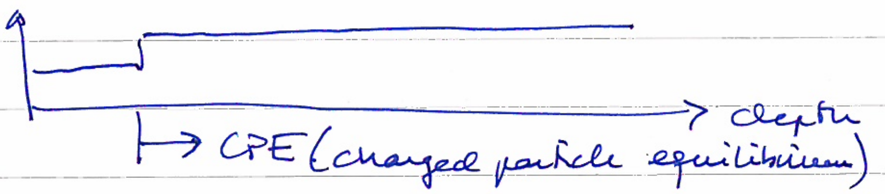
$$K = \Psi \left(\frac{\mu_{tr}}{\rho} \right)$$ where $\Psi = h\nu \cdot \Phi$ is the energy fluence of the photon beam.

Collision kerma : $$K_c = \Psi \left(\frac{\mu_{en}}{\rho} \right) = K(1-g)$$

b) Electron equilibrium at a certain depth in a given medium means that the dose contribution transported into deeper parts of the medium due to the forward direction of secondary electrons, is exactly compensated by dose deposits due to electrons coming into the given depth from more shallow positions.



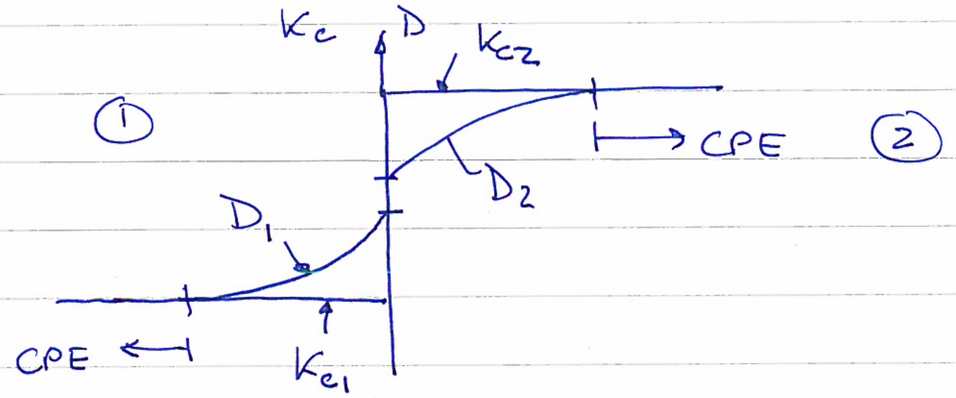
Relative ionization ($\propto$ dose):



CPE will not exist close to the surface of the medium.

Two media ① and ② where

$$\left(\frac{\mu_{en}}{\rho}\right)_1 < \left(\frac{\mu_{en}}{\rho}\right)_2 \quad \text{and} \quad \left(\frac{\bar{S}_c}{\rho}\right)_1 < \left(\frac{\bar{S}_c}{\rho}\right)_2 \, :$$



Negligible attenuation : $\Psi_1 = \Psi_2$

$$\Rightarrow \frac{K_{c2}}{K_{c1}} = \frac{\left(\frac{\mu_{en}}{\rho}\right)_2}{\left(\frac{\mu_{en}}{\rho}\right)_1} > 1 \text{ since } \Psi_i = \frac{K_{ci}}{\left(\frac{\mu_{en}}{\rho}\right)_i}$$

therefore there will be a step increase in K_c at the interface.

At the interface, the fluence of secondary electrons is continuous, i.e. Φ(left side of interface) = Φ(right side of interface).

$$\Rightarrow \text{At the interface} \quad \frac{D_2}{D_1} = \frac{\left(\frac{\bar{S}_c}{\rho}\right)_2}{\left(\frac{\bar{S}_c}{\rho}\right)_1} > 1. \text{ Since } D = \Phi\left(\frac{\bar{S}_c}{\rho}\right)$$

In material ① backscattered secondary electrons from medium ② will lead to a gradual increase in dose, since more secondary electrons are generated in medium ② $\left(\frac{\mu_{en}}{\rho}\right)_2 > \left(\frac{\mu_{en}}{\rho}\right)_1$

Dose in medium ② will increase gradually due to net forward transport of secondary electrons.

Discontinuity at the interface due to $\left(\frac{\bar{S}_c}{\rho}\right)_2 \neq \left(\frac{\bar{S}_c}{\rho}\right)_1$.

Lack of CPE close to the interface, as indicated.