TFY4225 Nuclear and Radiation Physics: Exam 2016 Suggested Solutions

Pål Erik Goa

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Problem 1

The key ingredient in the nuclear shell model was the spin-orbit coupling. The main potential function in the shell model consists of a central-symmetric part, but it was not until a small additional term consisting of a $\vec{l} \cdot \vec{s}$-term that the model was successful. The spin-orbit coupling results in a splitting of the original energy-levels as determined from l and s into new levels based on the new quantum number for total angular momentum j.

The main success of the nuclear shell model was twofold:

- Its ability to predict the correct "magic numbers", which are the neutron and proton numbers for particularly stable isotopes.
- Its ability to predict the correct spin and parity for the ground state of most odd-A nuclides. The model can also be used to predict spin-parity of possible excited states.

The "extreme independent particle model" is when we assume that the spin-parity state of the single unpaired nucleon in a odd-A nucleus determines the spin-parity for the nucleus as a whole. All other nucleons are paired and contributes with 0^+.

One example of when the "extreme independent particle model" does not work is the excited states of intermediately sized nuclides of even-even type. For these nuclides, a better model for the excited states is the liquid drop model with vibrational and/or rotational states.

Problem 2a

When calculating the Q-value for a nuclear reaction (nucleon number conserved) we can use directly the mass excess values:

$$Q = \sum_i m_i c^2 - \sum_f m_f c^2$$

$$= (3074\mu u + 7825\mu u - 2603\mu u - 11434\mu u) \cdot 931.5\ MeV/uc^2 \cdot c^2$$
$$= -2.923\ MeV$$

Due to the mass changes involved in a nuclear reaction, the expression for the kinetic energy T_b for the product particle b is quite complicated (see the formula sheet). For negative Q, the expression inside the square-root sign can become negative, and we end up with physically impossible solutions for certain values of T_a. This means that there exists a minimum threshold $T_a = T_{th}$ for which the reaction is possible (real-valued T_b). This is found by setting the expression inside the square-root sign to zero and choose $\theta = 0$:

$$\begin{aligned} m_a m_b T_{a,min} + (m_Y + m_b)\left[m_Y Q + (m_Y - m_a)T_{a,min}\right] &= 0 \\ T_{th} = T_{a,min} &= -Q\frac{m_Y + m_b}{m_Y + m_b - m_a} \end{aligned}$$

Sufficiently accurate estimate for the numerical value for T_{th} is found

$$\begin{aligned} T_{th} &= 2.923MeV\frac{11+4}{11+4-1} \\ &= \frac{15}{14} \cdot 2.923MeV = 3.132MeV \end{aligned}$$

Problem 2b

We calculate the value for the Coulomb potential energy P for the case of closest "hard-sphere" distance of the two nuclei:

$$\begin{aligned} R &= R_p + R_{^{14}N} \\ &= R_0(1 + 14^{1/3}) = 1.6\ fm \cdot 3.41 = 5.46\ fm \\ P &= \frac{q_1 q_2}{4\pi\epsilon_0 R} \\ &= 1.44\ MeV fm \cdot \frac{7 \cdot 1}{5.46\ fm} \\ &= 1.85MeV \end{aligned}$$

From figure 1 we can observe that the reaction cross section is zero for T_a below approximately 3.5 MeV. This is around the value we found for T_{th}. The cross section stays low until around 5 MeV where it rises sharply. The sum of the threshold energy and the Coulomb barrier as calculated in problem 2a and 2b is close to 5.0 MeV, and although some tunneling occur for T_a below that value, it is natural that the cross section rises when the total kinetic energy equals the sum of T_{th} and P.

Problem 2c

$$dN_1 = Rdt - \lambda_1 N_1 dt$$

To show that the expression given on the formula sheet is indeed the solution we plug the solution into the equation above and compare it with the direct dN_1/dt as given from the solution:

$$\begin{aligned}
N_1(t) &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\
\frac{dN_1}{dt} &= R - \lambda_1 N_1 = R - R(1 - e^{-\lambda_1 t}) \\
&= Re^{-\lambda_1 t} \\
\frac{dN_1}{dt} &= \frac{R}{\lambda_1}(0 - (-\lambda_1 e^{-\lambda_1 t})) \\
&= Re^{-\lambda_1 t}
\end{aligned}$$

The two expressions agree and we have shown that the expression in the formula sheet is the correct solution. Finally we calculate the required time to reach 90 % of maximum activity:

$$\begin{aligned}
A(t) &\equiv \lambda_1 N(t) \\
A(t_{0.9}) &= \lambda_1 \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t_{0.9}}) = 0.9R \\
t_{0.9} &= \frac{ln(0.1)}{-\lambda_1} = \frac{ln(10) t_{1/2}}{ln(2)} \\
&= 4.058 \cdot 10^3 \ s = 67.64 \ min
\end{aligned}$$

Problem 2d

The Fermi theory of beta decay uses Fermis Golden rule, which states that the decay rate is the product of the square of the matrix element V_{fi} between the initial and final quasi-stationary states of the system and the density of final states. The total final wavefunction for the system (which goes into the matric element) consists of the product of the nuclear wave function ψ_f and the electron and neutrino wave functions ϕ_e and ϕ_ν. The allowed approximation states that to the lowest order, the electron and neutrino wave functions, which in general has the free particle form $e^{i\vec{p}\cdot\vec{r}/\hbar}$, can be approximated by a constant value. This can be formally written as a Taylor expansion of the exponential function and keeping only the first term:

$$\begin{aligned}
e^{i\vec{p}\cdot\vec{r}/\hbar} &= 1 + \frac{i\vec{p}\cdot\vec{r}}{\hbar} + ... \\
&\approx 1
\end{aligned}$$

The approximation is justified by showing that the de-Broglie wavelength λ of the emitted electron is much larger than the size of the nucleus. For $T_e = 1$ MeV we get (using the relativistic expressions from the formula sheet):

$$E^2 = p^2c^2 + m_e^2 c^4 = (T_e + m_e c^2)^2$$

$$
\begin{aligned}
p &= \sqrt{\frac{T_e^2}{c^2} + 2T_e m_e} \\
&= 1.422 MeV/c \\
\lambda &= \frac{h}{p} = \frac{4.1357 \cdot 10^{-15} eV \cdot s \cdot 2.998 \cdot 10^8 m/s}{1.422 MeV} \\
&= 8.719 \cdot 10^{-13}\ m = 872\ fm
\end{aligned}
$$

The nuclear radius is in the order of a few fm, so in the overlap region between the nuclear and electron wave function, the electron wave function can be well approximated by a constant value.

In the allowed approximation, the two emitted particles can NOT carry away any orbital angular momentum (wave function centered on the origin, s-wave, l=0). However, both particles have spin 1/2, which can either be aligned in opposite directions (Fermi-type) or in the same direction (Gamow-Teller type). The above gives the following selection rules for the allowed decay (using usual vector addition for angular momentum conservation and that the parity of s-wave is even):

$$
\begin{aligned}
Fermi-type &: |I_i - I_f| = 0, \Delta\pi = \ no \\
G-T\ type &: 1 \geq |I_i - I_f| \geq 0, \Delta\pi = \ no
\end{aligned}
$$

A special case occurs for $I_i = I_f = 0$, for which only Fermi type decay is allowed.

The beta-decay of ^{11}C is a transition between a 3/2- state and a 3/2- state. This means that $|I_i - I_f| = 0$ and $\Delta\pi = no$. The decay is a combined Fermi-type and G-T type allowed transition.

Problem 2e

From the formula sheet we find the expression for the whole body effective dose. The given problem is very similar to problem 4 in week 10 exercises, and a more detailed discussion can be found there. Here we simply redo the numerical calculations with the current data:

$$
\begin{aligned}
\tilde{A}_{body} &= \int_0^{50\ years} A_0 e^{-\lambda t} dt \\
&\approx \int_0^{infty} A_0 e^{-\lambda t} dt = \frac{A_0}{\lambda} \\
&= \frac{A_0 T_{1/2}}{ln(2)} \frac{1 \cdot 10^9\ Bq \cdot 1.2217 \cdot 10^3\ s}{ln(2)} \\
&= 1.7625 \cdot 10^{12}
\end{aligned}
$$

$$
\begin{aligned}
S(r_T \leftarrow r_S) &= \frac{1}{M_T}\left[Q_{\beta+}/3 \cdot 1 \cdot \delta_{ST} + 2 \cdot 511\ keV \frac{kM_T}{M_{body}}\right] \\
&= \frac{320 keV}{M_T}\delta_{ST} + \frac{511 keV}{M_{body}}
\end{aligned}
$$

where we have set the photon absorbed fraction (k) to 0.5. This is a ball park value based on the attenuation length of 36 cm in soft tissue. The precise value is not critical, but to take the partial absorption of the annihilation photons into account is important.

$$\begin{aligned} E &= \sum_T w_T \sum_R w_R \sum_{r_S} \tilde{A}_{body} \frac{M_S}{M_{body}} \left[\frac{320keV}{M_T} \delta_{ST} + \frac{511keV}{M_{body}} \right] \\ &= \frac{\tilde{A}_{body}}{M_{body}} \sum_T w_T \sum_R w_R \left[205keV + \frac{496keV}{M_{body}} \sum_{r_S} M_S \right] \\ &= \frac{831keV \cdot \tilde{A}_{body}}{M_{body}} = 3.1mSv \end{aligned}$$

where we have used $M_{body} = 75$ kg, that $\sum_T w_T = 1$ and $\sum_S M_S = M_{body}$, and finally that w_R= 1 for both positrons and photons.

Problem 3

Energy imparted:

$$\begin{aligned} \varepsilon &= E_\gamma - E'_\gamma - E_b \\ &= 662keV - (662\ keV - 100\ keV) - 20\ keV = 80\ keV \end{aligned}$$

KERMA:

$$\begin{aligned} K &= \frac{dE_{tr}}{dm} = \frac{100keV}{1\mu g} \\ &= 1.6 \cdot 10^{-5} J/kg = 0.016\ mGy \end{aligned}$$

Problem 4 and 5

These two problems are very descriptive, and you are referred to the course literature.

Problem 6

The intention of this problem is not to perform any accurate calculation, but to make a qualitative argument over the relative differences in scattering and absorption cross sections of light and heavy water. The goal is to understand why heavy water is a better choice as moderator material than light water.

The requirement for a neutron emitted as result of fission to give rise to a new fission reaction, is that it must be absorbed by the fuel and NOT by any of the non-fissile nuclei in the reactor. Since the fuel absorption cross section has a 1/v dependence (highest probability of fission for low energy neutrons), the task is to bring the initially fast neutrons down in energy to the thermal range by means of scattering. This is the role of the moderator. The perfect moderator should have high scattering cross section and zero absorption cross section.

A higher scattering cross section will give faster neutron slow-down and therefore increase what is called the resonance escape probability p. One cannot predict directly the effect on p from the information given in the problem, but the scattering cross section of light water is only a factor 5 higher than that of heavy water, while the absorption cross section of light water is 660 times higher than heavy water. Qualitatively we can therefore argue that the moderate increase in resonance escape probability due to higher scattering cross section in light water will be more than cancelled by the massive increase in absorption cross section (which reduces the thermal utilization factor). Hence heavy water will give the highest neutron multiplication factor.