Problem 1

Consider a biased random walk in two dimensions. The displacement after N steps is: $\vec{R}_N = \sum_{n=1}^{N} \vec{\Delta}_n$, where $\vec{\Delta}_n = (1,0)$ with probability p and $\vec{\Delta}_n = (0,1)$ with probability $1-p$. There is no correlation between the steps, i.e. we have $\langle \vec{\Delta}_n \cdot \vec{\Delta}_m \rangle = \langle \vec{\Delta}_n \rangle \cdot \langle \vec{\Delta}_m \rangle$ whenever $n \neq m$. The mean square displacement after N steps is:

Select one alternative:

- $\langle \vec{R}_N^2 \rangle = Np(1-p)$
- $\langle \vec{R}_N^2 \rangle = N + (N^2 - N)\left(p^2 + (1-p)^2\right)$
- $\langle \vec{R}_N^2 \rangle = Np(1-p) + N^2(p^2 + (1-p)^2)$
- $\langle \vec{R}_N^2 \rangle = N^2(p^2 + (1-p)^2)$
- $\langle \vec{R}_N^2 \rangle = Np$
- $\langle \vec{R}_N^2 \rangle = N(p^2 + (1-p)^2)$

Solution 1

$$\begin{aligned} \langle \vec{\Delta}_i^2 \rangle &= 1 \\ \langle \vec{\Delta}_i \cdot \vec{\Delta}_j \rangle &= p^2 + (1-p)^2 \quad \text{for } i \neq j \end{aligned}$$

$$\langle \vec{R}_N^2 \rangle = \left\langle \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{\Delta}_i \cdot \vec{\Delta}_j \right\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} \left\langle \vec{\Delta}_i \cdot \vec{\Delta}_j \right\rangle = N + (N^2 - N)\left[p^2 + (1-p)^2\right]$$

Problem 2

Consider a particle in one dimension. The total energy is: $E = \frac{p_x^2}{2m} + K|x|$, where $K > 0$ is a constant, $-\infty < x < \infty$ is the position, and p_x the momentum. We are in the microcanonical ensemble where the total energy is constant, and the probability distribution is: $P(p_x, x) = C\,\delta\left[\frac{p_x^2}{2m} + K|x| - E\right]$ where C is a constant. Energy conservation restricts the momentum to the interval: $-\sqrt{2mE} \leq p_x \leq \sqrt{2mE}$. The constant C is equal to:

Select one alternative:

- ○ $C = \frac{K}{2mE}$
- ○ $C = 1$
- ○ $C = \frac{K}{4\sqrt{2mE}}$
- ○ $C = K$
- ○ $C = \frac{K^2}{\sqrt{2mE}}$
- ○ $C = \frac{K}{m^2E^2}$

Solution 2

$$\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \mathrm{d}p_x \int_{-\infty}^{\infty} \mathrm{d}x\, C\,\delta[\frac{p_x^2}{2m} - E + K|x|] = \int_{-\sqrt{2mE}}^{\sqrt{2mE}} \mathrm{d}p_x \frac{2}{K} = \frac{4C\sqrt{2mE}}{K} \Rightarrow C = \frac{K}{4\sqrt{2mE}}$$

Problem 3

Consider a microcanonical system. The entropy as a function of energy is $S = C\ln(E)$, where C is a constant. The relation between energy and temperature is:

Select one alternative:

- ○ $E = C\ln(T)$
- ○ $E = k_BT$
- ○ $E = \ln(CT)$
- ○ $E = Ck_BT$
- ○ $E = CT$
- ○ $E = \frac{C}{T}$

Solution 3

$$\frac{dS}{dE} = \frac{1}{T} \Rightarrow E = CT$$

Problem 4

Consider a particle in two dimensions with Hamiltonian: $H = \frac{p_x^2+p_y^2}{2m} + U(x,y)$. We are in the canonical ensemble. The potential energy is $U(x,y) = K|x| + K|y|$, where K is a positive constant. The particle can take any position on the plane: i.e. we have : $-\infty < x < \infty$ and $-\infty < y < \infty$. The partition function is:

Select one alternative:

- ○ $Z = \frac{2\pi m}{h^2}(k_B T)^2 K^2$
- ○ $Z = \frac{h^2}{2\pi m}(k_B T)^2 \mathrm{e}^{-2\frac{K}{k_B T}}$
- ○ $Z = \frac{2\pi m}{h^2} k_B T \frac{1}{K}$
- ○ $Z = \frac{2\pi m}{h^2}(k_B T)^3 \frac{4}{K^2}$
- ○ $Z = \frac{2\pi m}{h^2} k_B T \mathrm{e}^{-2\frac{K}{k_B T}}$
- ○ $Z = \frac{h^2}{2\pi m} k_B T \mathrm{e}^{-2\frac{K^2}{k_B T}}$

Solution 4

$$Z = \frac{1}{h^2}\int d^2p \mathrm{e}^{-\beta\frac{p^2}{2m}} \int d^2r \mathrm{e}^{-\beta U} = \frac{1}{h^2}\left(\int_{-\infty}^{\infty} dp_x \mathrm{e}^{-\beta\frac{p_x^2}{2m}}\right)^2 \left(\int_{-\infty}^{\infty} dx \mathrm{e}^{-\beta K|x|}\right)^2 = \frac{2\pi m}{h^2}(k_B T)^3 \frac{4}{K^2}$$

Problem 5

A system with N particles has the density of states: $\rho(E) = C^N$ with $E \geq 0$ and C a positive constant. We are in the canonical ensemble and the partition function is $Z = \frac{1}{N! h^{3N}} \int_0^\infty dE\, \rho(E) e^{-\beta E}$. The mean energy is:

Select one alternative:

- ○ $\langle E \rangle = N k_B T$
- ○ $\langle E \rangle = N k_B T + k_B T \ln(C)$
- ○ $\langle E \rangle = \frac{N}{2} k_B T$
- ○ $\langle E \rangle = NC$
- ○ $\langle E \rangle = N \ln(C) k_B T$
- ○ $\langle E \rangle = k_B T$

Solution 5

$$Z = \frac{1}{N!h^{3N}} \int_0^\infty dE\, C^N e^{-\beta E} = \frac{C^N}{N!h^{3N}} \frac{1}{\beta}$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = \frac{1}{\beta} = k_B T$$

Problem 6

Consider a particle in one dimension, confined to the interval $0 < x < L$. The Hamiltonian is $H = \frac{p_x^2}{2m} + U(x)$, where $U(x) = \epsilon$ when $0 < x < \ell$ and $U(x) = 0$ when $\ell < x < L$. We are in the canonical ensemble. Keep ℓ constant and finite and let $L \to \infty$. What is the mean potential energy?

Select one alternative:

- ○ $\langle U \rangle = \infty$
- ○ $\langle U \rangle = 0$
- ○ $\langle U \rangle = k_B T$
- ○ $\langle U \rangle = \frac{(k_B T)^2}{\epsilon}$
- ○ $\langle U \rangle = \epsilon$
- ○ $\langle U \rangle = \frac{\epsilon^2}{k_B T}$

Solution 6

$$Z = \frac{1}{h} \int dp_x \int dx\, \mathrm{e}^{-\beta H} = \frac{1}{h} \left(\frac{2\pi m}{\beta} \right)^{1/2} \left[\ell \mathrm{e}^{-\beta\epsilon} + (L - \ell) \right]$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = \frac{1}{2} k_B T + \frac{\ell \epsilon}{\ell + (L - l) \mathrm{e}^{\beta\epsilon}}$$

$$\langle U \rangle = \frac{\ell \epsilon}{\ell + (L - l) \mathrm{e}^{\beta\epsilon}} \to 0 \text{ when } L \to \infty$$

Problem 7

Consider a system with energy levels $E = n\epsilon$, with $n = 0, 1, 2, \cdots, \infty$ and ϵ is a positive constant. We are in the canonical ensemble. What is the probability p_2 that the system is in the state $n = 2$?

Select one alternative:

- ○ $p_2 = \frac{e^{-2\beta\epsilon}}{1+e^{\beta\epsilon}}$
- ○ $p_2 = \frac{e^{-2\beta\epsilon}}{1-e^{-\beta\epsilon}}$
- ○ $p_2 = e^{-2\beta\epsilon}$
- ○ $p_2 = e^{-2\beta\epsilon} - e^{-4\beta\epsilon}$
- ○ $p_2 = e^{-\beta\epsilon} + e^{-2\beta\epsilon}$
- ○ $p_2 = e^{-2\beta\epsilon} - e^{-3\beta\epsilon}$

Solution 7

$$Z = \sum_{n=0}^{\infty} e^{-\beta n\epsilon} = \frac{1}{1 - e^{-\beta\epsilon}}$$

$$p_2 = \frac{e^{-2\beta\epsilon}}{Z} = e^{-2\beta\epsilon} - e^{-3\beta\epsilon}$$

Problem 8

Consider two harmonic oscillators with Hamiltonians: $H_1 = \frac{p_x^2}{2m_1} + \frac{1}{2}m_1\omega^2x^2$ and $H_2 = \frac{p_x^2}{2m_2} + \frac{1}{2}m_2\omega^2x^2$. Both harmonic oscillators have the same constant energy ($E > 0$), and $m_1 = 4m_2$. Which statements is correct?

Select one alternative:

- ○ The phase trajectories of the oscillators cross in four points.
- ○ The phase trajectories of the two the oscillators cannot cross.
- ○ The phase trajectories of the oscillators can only cross if the energy is larger than a critical value.
- ○ The phase trajectories of the two oscillators cross in infinitely may points
- ○ The phase trajectories of the two oscillators are identical.
- ○ The phase trajectories of both oscillators are infinitely long.

Solution 8 The phase space trajectory of a harmonic oscillator is an ellipse in the (x, p_x) space with $-\sqrt{2mE} \le p_x \le \sqrt{2mE}$ and $-\sqrt{\frac{2E}{m\omega^2}} \le x \le \sqrt{\frac{2E}{m\omega^2}}$. Since the two oscillators have different different mass the two ellipses cross in 4 points.

Problem 9

Consider two particles in one dimension. The Hamiltonian is: $H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + K(x_1 - x_2)^2$, with $K > 0$.
We are in the canonical ensemble, the mean square distance between the particles is:

Select one alternative:

- ○ $\langle (x_1 - x_2)^2 \rangle = \frac{k_B T}{8K}$
- ○ $\langle (x_1 - x_2)^2 \rangle = \frac{k_B T}{2K}$
- ○ $\langle (x_1 - x_2)^2 \rangle = \frac{k_B T}{2}$
- ○ $\langle (x_1 - x_2)^2 \rangle = \frac{2k_B T}{K}$
- ○ $\langle (x_1 - x_2)^2 \rangle = \frac{k_B T}{2} \frac{m_1 m_2}{K^2}$
- ○ $\langle (x_1 - x_2)^2 \rangle = \frac{k_B T}{4K}$

Solution 9

$$Z = \frac{1}{h^2 2!} \int dp_x \int dp_y \int dx_1 \int dx_2 \, \mathrm{e}^{-\beta[\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + K(x_1 - x_2)^2]}$$

$$\langle (x_1 - x_2)^2 \rangle = -\frac{\partial}{\beta \partial K} \ln Z$$

$$\langle (x_1 - x_2)^2 \rangle = -\frac{\partial}{\beta \partial K} \ln \left[\int dx_1 \int dx_2 \, \mathrm{e}^{-\beta K (x_1 - x_2)^2} \right] = -\frac{\partial}{\beta \partial K} \ln \sqrt{\frac{\pi}{\beta K}}$$

$$\langle (x_1 - x_2)^2 \rangle = \frac{k_B T}{2K}$$

Problem 10

Consider an ideal two-dimensional gas of N classical particles. The particles are confined to the surface defined by $0 < x < L$ and $0 < y < \infty$. The particles are subject to a gravitational field and each particle has the potential energy $U = mgy$. We are in the canonical ensemble. What is the probability p that the y coordinate of all the particles are on the interval $0 < y < L$?

Select one alternative:

- ○ $p = \left[1 - \mathrm{e}^{-\beta mgL}\right]^N$
- ○ $p = \frac{1}{L^N}$
- ○ $p = \left[1 - \frac{1}{L}\right]^N$
- ○ $p = \frac{1}{2^N}$
- ○ $p = \frac{1}{1+\mathrm{e}^{-\beta mgL}}$
- ○ $p = \mathrm{e}^{-N\beta mgL}$

Solution 10 Probability that one particle is on the prescribed interval:

$$p_1 = \int_0^L P(y)\mathrm{d}y$$

where the Boltzmann distribution is $P(y) = C\mathrm{e}^{-\beta U(y)}$, normalization of the probability distribution determines the constant C, which gives $P(y) = mg\beta\mathrm{e}^{-\beta mgy}$ and

$$p_1 = 1 - \mathrm{e}^{-\beta mgL}$$

hence the probabilty that all particles is in the interval is

$$p = p_1^N = \left[1 - \mathrm{e}^{-\beta mgL}\right]^N$$

Problem 11

A system of N classical distinguishable particles can occupy three energy levels, one ground state and two excited states. How many different configurations are there with 1 particle in the ground state?

Select one alternative:

- ○ $\frac{N(N-1)(N-2)}{3}$
- ○ 2^N
- ○ $N\,2^{N-1}$
- ○ N
- ○ $N(N-1)\,2^N$
- ○ N^2

Solution 11 N ways of choosing particle for the ground state. The remaining $N-1$ particles are distributed on two energy leves, giving the total number of configurations $N \times 2^{N-1}$

Problem 12

A system with N particles has the free energy: $F = N\epsilon - k_B T \ln(N)$. We are in the canonical ensemble. The variance of the energy $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$ is:

Select one alternative:

- ○ $\Delta E^2 = \frac{\epsilon^2}{N}$
- ○ $\Delta E^2 = N k_B T \epsilon$
- ○ $\Delta E^2 = N\epsilon^2$
- ○ $\Delta E^2 = 0$
- ○ $\Delta E^2 = k_B T \epsilon$
- ○ $\Delta E^2 = (k_B T)^2$

Solution 12

$$F = N\epsilon - k_B T \ln(N) \Rightarrow Z = \mathrm{e}^{-\beta F} = N\mathrm{e}^{-\beta N\epsilon}$$

The system has only one energy level $E = N\epsilon$, and hence $\Delta E^2 = 0$.

Problem 13

Consider a system of two spins with Hamiltonian: $H = -J s_1 s_2 - h s_1 + h s_2$, where J and h are positive constant, and $s_i = \pm 1$. We are in the canonical ensemble. Suppose that $h > J$, what is the mean energy $\langle E \rangle$ when the temperature goes to zero $T \to 0$?

Select one alternative:

- ○ $\langle E \rangle = -h$
- ○ $\langle E \rangle = J - 2h$
- ○ $\langle E \rangle = -J - 2h$
- ○ $\langle E \rangle = -J$
- ○ $\langle E \rangle = 0$
- ○ $\langle E \rangle = -2h$

Solution 13

There are four states of the spins $(s_1, s_2) = (1,1), (1,-1), (-1,1), (-1,-1)$ with corresponding energies $H = -J, J-2h, J+2h, -J$. Since $h > 2J$ the lowest energy is $H = J - 2h$ which is the ground state at zero temperature, i.e. $\langle E \rangle = J - 2h$ at $T = 0$.

Problem 14

Consider a system of two spins with Hamiltonian: $H = -Js_1 s_2 + Js_1 - Js_2$, where J is a positive constant, and $s_i = \pm 1$. We are in the canonical ensemble. What is the entropy S of the system when the temperature goes to zero $T \to 0$?

Select one alternative:

- ○ $S = k_B J$
- ○ $S = k_B \ln(8)$
- ○ $S = k_B \ln(3)$
- ○ $S = 2k_B$
- ○ $S = 0$
- ○ $S = 3k_B$

Solution 14 There are four states of the spins $(s_1, s_2) = (1,1), (1,-1), (-1,1), (-1,-1)$ with corresponding energies $H = -J, 3J, -J, -J$. At $T = 0$ there are therefore 3 ground states with lowest energy $H = -J$, and the entropy is $S = k_B \ln 3$.

Problem 15

A system is described by the Hamiltonian $H = J(s_1 s_2 + s_2 s_3 + s_3 s_1)$ where $J > 0$ and the spins take the values $s_i = \pm 1$. When $T \to \infty$ the mean energy and entropy become:

Select one alternative:

- ○ $\langle E \rangle = -3J$ and $S = k_B 3 \ln(2)$
- ○ $\langle E \rangle = -J$ and $S = k_B 3 \ln(2)$
- ○ $\langle E \rangle = 0$ and $S = 3k_B$
- ○ $\langle E \rangle = 0$ and $S = k_B 3 \ln(2)$
- ○ $\langle E \rangle = 0$ and $S = k_B \ln(3)$
- ○ $\langle E \rangle = 0$ and $S = 0$

Solution 15 At infinite temperature there is no correlation between the spins and $\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle = 0$ when $i \neq j$, hence $\langle H \rangle = 0$. There are 2^3 configurations of the spins hence:

$$\langle E \rangle = 0 \text{ and } S = k_B \ln(8) = k_B 3 \ln(2)$$

Problem 16

Consider a system with Hamiltonian $H = -\epsilon(s_1 + s_1 s_2 + s_1 s_2 s_3)$ with spins that can take the values $s_i = \pm 1$. We are in the canonical ensemble. The partition function is:

Select one alternative:

- $Z = e^{3\beta\epsilon} + 2e^{2\beta\epsilon} + 2e^{-2\beta\epsilon} + e^{-3\beta\epsilon}$
- $Z = e^{3\beta\epsilon} + 3e^{2\beta\epsilon} + 3e^{\beta\epsilon} + 1$
- $Z = 2e^{3\beta\epsilon} + 2e^{\beta\epsilon} + 2e^{-\beta\epsilon} + 2e^{-3\beta\epsilon}$
- $Z = e^{4\beta\epsilon} + 3e^{2\beta\epsilon} + 3e^{-2\beta\epsilon} + e^{-4\beta\epsilon}$
- $Z = e^{3\beta\epsilon} + e^{-3\beta\epsilon}$
- $Z = e^{3\beta\epsilon} + 3e^{\beta\epsilon} + 3e^{-\beta\epsilon} + e^{-3\beta\epsilon}$

Solution 16

$$
\begin{aligned}
Z &= \sum_{s_1=\pm1}\sum_{s_2=\pm1}\sum_{s_3=\pm1} e^{\beta\epsilon(s_1+s_1s_2+s_1s_2s_3)} \\
&= \sum_{s_1=\pm1}\sum_{s_2=\pm1} e^{\beta\epsilon(s_1+s_1s_2)}(e^{\beta\epsilon s_1s_2} + e^{-\beta\epsilon s_1s_2}) \\
&= \sum_{s_1=\pm1}\sum_{s_2=\pm1} e^{\beta\epsilon(s_1+s_1s_2)}(e^{\beta\epsilon} + e^{-\beta\epsilon}) \\
&= \sum_{s_1=\pm1} e^{\beta\epsilon s_1}(e^{\beta\epsilon} + e^{-\beta\epsilon})^2 \\
&= (e^{\beta\epsilon} + e^{-\beta\epsilon})^3
\end{aligned}
$$

$$
Z = e^{3\beta\epsilon} + 3e^{\beta\epsilon} + 3e^{-\beta\epsilon} + e^{-3\beta\epsilon}
$$

Problem 17

A system has the Hamiltonian $H = -J\phi_1\phi_2$, where $J > 0$ is a constant and the variables ϕ_i can take the two values $\phi_i = 0, 1$. We are in the canonical ensemble. The average energy is:

Select one alternative:

- ○ $\langle E \rangle = -J\mathrm{e}^{2\beta J}$
- ○ $\langle E \rangle = -J\mathrm{e}^{-\beta J}$
- ○ $\langle E \rangle = -J$
- ○ $\langle E \rangle = -\frac{J}{1+3\mathrm{e}^{-\beta J}}$
- ○ $\langle E \rangle = -J\frac{\mathrm{e}^{2\beta J}+\mathrm{e}^{-2\beta J}}{1+\mathrm{e}^{2\beta J}+\mathrm{e}^{-2\beta J}}$
- ○ $\langle E \rangle = -J\frac{\mathrm{e}^{\beta J}}{\mathrm{e}^{\beta J}+\mathrm{e}^{-\beta J}}$

Solution 17

$$Z = \sum_{\phi_1=0,1}\sum_{\phi_2=0,1} \mathrm{e}^{\beta J\phi_1\phi_2} = 3 + \mathrm{e}^{\beta J}$$

$$\langle E \rangle = -\frac{J}{1+3\mathrm{e}^{-\beta J}}$$

Problem 18

A system is given by the Hamiltonian:

$H = \sum_{i=1}^{N}(-hq_i + \epsilon q_i^2) = -h(q_1 + q_2 + \cdots + q_N) + \epsilon(q_1^2 + s_2^2 + \cdots + q_N^2)$ where the variables can take any real value: $-\infty < q_i < \infty$, and h and ϵ are positive constants. We are in the canonical ensemble. When $T = 0$ the average energy is:

Select one alternative:

- ○ $\langle E \rangle = -N\epsilon$
- ○ $\langle E \rangle = -N\sqrt{h\epsilon}$
- ○ $\langle E \rangle = -N\frac{h}{\epsilon}$
- ○ $\langle E \rangle = -N\frac{h^2}{4\epsilon}$
- ○ $\langle E \rangle = -N\frac{h}{\epsilon}$
- ○ $\langle E \rangle = -N^2\frac{h^2}{\epsilon^2}$

Solution 18 At $T = 0$ the average energy is the minimum energy of the Hamiltonian. The minimum of $-hq_1 + \epsilon q_1^2$ is $-\frac{h^2}{4\epsilon}$, hence:

$$\langle E \rangle = -N\frac{h^2}{4\epsilon}$$

Problem 19

Consider a spin system with the Hamiltonian: $H = J\vec{s}_1 \cdot \vec{s}_2$ where $J > 0$, $\vec{s}_i = (\cos\theta_i, \sin\theta_i)$ and $-\pi \le \theta_i \le \pi$. We are in the canonical ensemble. In the low temperature limit $k_B T \ll J$ we have:

Select one alternative:

- ○ $\langle(\theta_1 - \theta_2)^2\rangle \approx \frac{\pi}{2}$
- ○ $\langle(\theta_1 - \theta_2)^2\rangle \approx \pi$
- ○ $\langle(\theta_1 - \theta_2)^2\rangle \approx 4\pi^2$
- ○ $\langle(\theta_1 - \theta_2)^2\rangle \approx 0$
- ○ $\langle(\theta_1 - \theta_2)^2\rangle \approx \pi^4$
- ○ $\langle(\theta_1 - \theta_2)^2\rangle \approx \pi^2$

Solution 19 The Hamiltonian favors anti-alignment of the spins, the minimum energy is when $\theta_1 - \theta_2 = \pm\pi$, hence $\langle(\theta_1 - \theta_2)^2\rangle \approx \pi^2$ at low temperatures.

Problem 20

Consider a system (S) that has free energy $F_S = -1000\epsilon N + \epsilon N^2$ where N is the number of particles in the system and $\epsilon > 0$ is a positive constant. The volume is constant. The system is connected to a reservoir of N_r particles with free energy $F_r = N_r\mu$, where $\mu > -1000\epsilon$ is a constant chemical potential. Assume that the number of particles in the reservoir is infinite $N_r = \infty$. What is the average number of particles N in the system S at thermal equilibrium?

Select one alternative:

- ○ $N = -\frac{\mu}{\epsilon}$ but only if $|\mu|$ is larger than some critical value
- ○ $N = 1000$
- ○ $N = 0$
- ○ $N = 500 + \frac{\mu}{2\epsilon}$
- ○ $N = \infty$
- ○ $N = 500$

Solution 20 Thermal equilibrium implies:

$$\frac{\partial F}{\partial N} = \mu \Rightarrow 1000\epsilon - 2\epsilon N = \mu$$

Hence:

$$N = 500 + \frac{\mu}{2\epsilon}$$

Problem 21

Consider a classical ideal gas in three dimensions confined to a volume V. Each particle carries a spin that is aligned by an external field. The Hamiltonian is $H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} - Bs_i$, where B is a positive constant and $s_i = \pm 1$. The thermal de Broglie length is λ.The chemical potential μ of the gas is:

Select one alternative:

- ○ $\mu = \frac{1}{\beta}\ln\left(\frac{N\lambda^3}{V}\right) - B$
- ○ $\mu = \frac{1}{\beta}\frac{N\lambda^3}{V} - \frac{N}{\beta}\ln\left(\mathrm{e}^{\beta B} + \mathrm{e}^{-\beta B}\right)$
- ○ $\mu = \frac{1}{\beta}\ln\left(\frac{N\lambda^3}{V}\right) - B\frac{V}{\lambda^3}$
- ○ $\mu = \frac{1}{\beta}\ln\left(\frac{N\lambda^3}{V}\right) - \frac{1}{\beta}\ln\left(\mathrm{e}^{\beta B} + \mathrm{e}^{-\beta B}\right)$
- ○ $\mu = \frac{1}{\beta}\ln\left(\frac{N\lambda^3}{V}\right) - \frac{1}{\beta}\left(\mathrm{e}^{\beta B} + \mathrm{e}^{-\beta B}\right)$
- ○ $\mu = \frac{1}{\beta}\frac{N\lambda^3}{V} - B\frac{V}{\lambda^3}$

Solution 21 Partition function:

$$Z = \frac{1}{h^{3N}N!}\left(\int \mathrm{d}^3p\mathrm{e}^{-\beta\frac{p^2}{2m}}\right)^N\left(\sum_{s_1=\pm 1}\mathrm{e}^{\beta B s_1}\right)^N = \frac{1}{h^{3N}N!}\left(\frac{2\pi m}{\beta}\right)^{3N/2}\left(\mathrm{e}^{\beta B} + \mathrm{e}^{-\beta B}\right)^N$$

Free energy:

$$F = -k_B T \ln Z$$

Chemical potential:

$$\mu = \frac{\partial F}{\partial N} = \frac{1}{\beta}\ln\left(\frac{N\lambda^3}{V}\right) - \frac{1}{\beta}\ln\left(\mathrm{e}^{\beta B} + \mathrm{e}^{-\beta B}\right)$$

Problem 22

A system of particles has a grand canonical partition function: $\Theta = \exp\left[\beta\mu + (\beta\mu)^2\right]$ where μ is the chemical potential. The particles are confined to a volume V. Assume that the average number of particles in the system is large: $\langle N \rangle \gg 1$. The pressure p in the system is:

Select one alternative:

- ○ $p \approx \frac{k_B T}{V}[\langle N \rangle^2 + \langle N \rangle^3]$
- ○ $p \approx \frac{k_B T}{V}\langle N \rangle^{-1}$
- ○ $p \approx \frac{k_B T}{V}[\langle N^2 \rangle - \langle N \rangle^2]$
- ○ $p \approx \frac{k_B T}{V}\frac{1}{4}\langle N \rangle^2$
- ○ $p \approx \frac{k_B T}{V}[\frac{1}{2}\langle N \rangle + \frac{1}{4}\langle N \rangle^4]$
- ○ $p \approx \frac{k_B T}{V}\langle N \rangle^2$

Solution 22

$$\langle N \rangle = \frac{\partial}{\beta \partial \mu} \ln \Theta = 1 + 2\beta\mu$$

$$p = \frac{k_B T}{V} \ln \Theta = \frac{k_B T}{V}(\beta\mu + \beta^2\mu^2) \approx \frac{k_B T}{V}\frac{1}{4}\langle N \rangle^2$$

Problem 23

A system of particles has the grand partition function: $\Theta = \exp\left(M\beta\mu - M^2\beta\epsilon\right)$ where $\epsilon > 0$ is a constant, and $M \gg 1$ a constant number, μ is the chemical potential and $\beta = \frac{1}{k_B T}$. Assume that the average number of particles is large: $\langle N \rangle \gg 1$. The average energy of the system can be expressed as:

Select one alternative:

- ○ $\langle E \rangle = [\langle N^2 \rangle - \langle N \rangle^2]\epsilon$
- ○ $\langle E \rangle = [\langle N \rangle + \langle N \rangle^2]\frac{1}{\beta}$
- ○ $\langle E \rangle = \langle N \rangle \frac{1}{\beta}$
- ○ $\langle E \rangle = \langle N \rangle^2 \epsilon$
- ○ $\langle E \rangle = \langle N \rangle \epsilon$
- ○ $\langle E \rangle = \epsilon$

Solution 23

$$\langle N \rangle = \frac{\partial}{\beta \partial \mu} \ln \Theta = M$$

$$\langle E\rangle = \mu\langle N\rangle - \frac{\partial}{\partial\beta}\ln\Theta = \langle N\rangle^2\epsilon$$

Problem 24

A system has the canonical partition function $Z_N = \frac{1}{N!}\mathrm{e}^{\beta\epsilon N}$, where ϵ is a constant, and N is the number of particles. The system is connected to a particle reservoir with chemical potential μ. Use the grand canonical partition function to calculate the fluctuations in particle number $\Delta N^2 = \langle N^2\rangle - \langle N^2\rangle$ when $\mu = 0$.

Select one alternative:

- $\Delta N^2 = \frac{1}{\beta\epsilon}$
- $\Delta N^2 = \mathrm{e}^{\beta\epsilon}$
- $\Delta N^2 = \beta\epsilon$
- $\Delta N^2 = 1$
- $\Delta N^2 = \mathrm{e}^{\beta\epsilon} + \mathrm{e}^{-\beta\epsilon}$
- $\Delta N^2 = \mathrm{e}^{-2\beta\epsilon}$

Solution 24

$$\Theta = \sum_{N=0}^{\infty} Z_N \mathrm{e}^{\beta\mu N} = \sum_{N=0}^{\infty}\frac{1}{N!}\mathrm{e}^{\beta(\mu+\epsilon)N} = \exp\left[\mathrm{e}^{\beta(\mu+\epsilon)}\right]$$

$$\langle N\rangle = \frac{\partial}{\beta\partial\mu}\ln\Theta = \mathrm{e}^{\beta(\mu+\epsilon)}$$

$$\Delta N^2 = \frac{\partial}{\beta\partial\mu}\langle N\rangle = \mathrm{e}^{\beta(\mu+\epsilon)}$$

When $\mu = 0$ we get: $\Delta N^2 = \mathrm{e}^{\beta\epsilon}$.

Problem 25

A system has the canonical partition function $Z_N = \sum_{n=0}^{N} e^{-\beta\epsilon n}$, where ϵ is a positive constant and $N > 1$ is a finite number. When the temperature goes to infinity ($T \to \infty$) the mean energy $\langle E \rangle$ becomes:

Select one alternative:

- ○ $\langle E \rangle = \frac{N+1}{2N}\epsilon$
- ○ $\langle E \rangle = \frac{N^2}{2}\epsilon$
- ○ $\langle E \rangle = \frac{N^2-N}{2}\epsilon$
- ○ $\langle E \rangle = \frac{1}{2}\epsilon$
- ○ $\langle E \rangle = \frac{N}{2}\epsilon$
- ○ $\langle E \rangle = \frac{N^2+1}{2}\epsilon$

Solution 25 The partition function tells us that there are $N + 1$ energy levels: $E_n = n\epsilon$ with $n = 0, 1, 2, \cdots, N$. At infinite temperature all energy levels have same probability of being occupied. Hence the average energy becomes

$$\langle E \rangle = \frac{1}{N+1} \sum_{n=0}^{N} E_n = \frac{1}{N+1} \frac{N(N+1)}{2} \epsilon$$

$$\langle E \rangle = \frac{N}{2} \epsilon$$

Problem 26

A system has the grand canonical partition function $\Theta = \sum_{N=0}^{\infty} Z_N e^{\beta\mu N}$, where $Z_N = \frac{1}{N!} e^{-\beta\epsilon N}$ is the canonical partition function, Ndenotes the number of particles, and ϵ is a posititive constant. When the chemical potential goes to minus infinity ($\mu \to -\infty$) the average number of particles in the system is:

Select one alternative:

- ○ $\langle N \rangle = \frac{1}{1+e^{-\beta\epsilon}}$
- ○ $\langle N \rangle = 1$
- ○ $\langle N \rangle = \infty$
- ○ $\langle N \rangle = \frac{1}{\beta\epsilon}$
- ○ $\langle N \rangle = 0$
- ○ $\langle N \rangle = \beta\epsilon$

Solution 26

$$\Theta = \sum_{N=0}^{\infty} Z_N e^{\beta\mu N} = \exp\left[e^{\beta(\mu-\epsilon)}\right]$$

$$\langle N\rangle = \frac{\partial}{\beta\partial\mu}\ln\Theta = e^{\beta(\mu-\epsilon)}$$

When $\mu \to -\infty$ we have:

$$\langle N\rangle = 0$$

Problem 27

Consider a 1D paramagnet with an external magnetic field: $H = -\mu B \sum_{i=1}^{N} s_i$ where μ is the magnetic moment, B the external field, and the spin can take the values $s_i = \pm 1$. We are in the canonical ensemble. In the low temperature limit the energy fluctuations in the system $\Delta E^2 = \langle E^2\rangle - \langle E\rangle^2$ is:

Select one alternative:

- ○ $\Delta E^2 \approx N\epsilon^2$
- ○ $\Delta E^2 \approx N(2\mu B)^2 e^{-2\beta\mu B}$
- ○ $\Delta E^2 \approx \sqrt{N}\beta\mu B$
- ○ $\Delta E^2 \approx N\frac{1}{\beta^2}$
- ○ $\Delta E^2 \approx N(2\beta\mu B)^2$
- ○ $\Delta E^2 \approx e^{-\beta\mu B}$

Solution 27

$$Z = \left(\sum_{s_1=\pm 1} e^{\beta\mu B s_1}\right)^N = \left(e^{\beta\mu B} + e^{-\beta\mu B}\right)^N$$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = -N\mu B\,\frac{e^{\beta\mu B} - e^{-\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}} = -N\mu B\,\frac{1 - e^{-2\beta\mu B}}{1 + e^{-2\beta\mu B}} \approx -N\mu B\left(1 - 2e^{-2\beta\mu B}\right)$$

$$\Delta E^2 = -\frac{\partial}{\partial\beta}\langle E\rangle \approx N(2\mu B)^2 e^{-2\beta\mu B}$$

Problem 28

A molecule can switch between three configurations (C_1, C_2, C_3). The energies of the configurations are $E_1 = 0$, $E_2 = \epsilon$, $E_3 - = -\epsilon$. We are in the canonical ensemble and the temperature T is such that the probabilities to be in the corresponding configurations are: $p_1 = \frac{2}{7}$, $p_2 = \frac{1}{7}$, $p_3 = \frac{4}{7}$. The free energy F is:

Select one alternative:

- ○ $F = k_BT \ln(7)$ with $k_BT = \frac{\epsilon}{2\ln(2)}$
- ○ $F = k_BT \ln(3)$ with $k_BT = \frac{\epsilon}{2\ln(2)}$
- ○ $F = k_BT \ln(\frac{2}{7})$ with $k_BT = \frac{\epsilon}{\ln(2)}$
- ○ $F = k_BT \ln(\frac{1}{7})$ with $k_BT = \frac{\epsilon}{2}$
- ○ $F = \frac{3}{2}k_BT$ with $k_BT = \epsilon$
- ○ $F = k_BT \ln(\frac{4}{7})$ with $k_BT = \frac{\epsilon}{7\ln(2)}$

Solution 28

Partition function:

$$Z = 1 + e^{\beta\epsilon} + e^{-\beta\epsilon}$$

Probabilities:

$$\begin{aligned} p_1 &= \frac{1}{Z} \\ p_2 &= \frac{e^{-\beta\epsilon}}{Z} \\ p_3 &= \frac{e^{\beta\epsilon}}{Z} \end{aligned}$$

The relation between free energy and partition function implies: $F = -k_BT \ln Z = k_BT \ln(p_1)$, i.e. we have:

$$F = k_BT \ln(\frac{2}{7}) \text{ with } k_BT = \frac{\epsilon}{\ln(2)}$$

Problem 29

In the Einstein model for the heat capacity of solids it is assumed that the density of states is $g(\omega) = 3N\delta(\omega - \omega_0)$. The mean energy is $\langle E\rangle = \int_0^\infty \frac{g(\omega)\hbar\omega\,\mathrm{d}\omega}{\mathrm{e}^{\beta\hbar\omega}-1}$. What is the heat capacity in the low temperature limit?

Select one alternative:

- ○ $C \approx 3Nk_B(\beta\hbar\omega_0)^4$
- ○ $C \approx 3Nk_B\mathrm{e}^{-\beta\hbar\omega_0}$
- ○ $C \approx 3Nk_B(\beta\hbar\omega_0)^2\mathrm{e}^{-\beta\hbar\omega_0}$
- ○ $C \approx 3Nk_B$
- ○ $C \approx 3Nk_B(\beta\hbar\omega_0)^4\mathrm{e}^{-\beta\hbar\omega_0}$
- ○ $C \approx 3Nk_B\mathrm{e}^{\beta\hbar\omega_0}$

Solution 29

$$\langle E\rangle = \int_0^\infty \frac{g(\omega)\hbar\omega\,\mathrm{d}\omega}{\mathrm{e}^{\beta\hbar\omega}-1} = \frac{3N\hbar\omega_0}{\mathrm{e}^{\beta\hbar\omega_0}-1}$$

When $\beta \to \infty$:

$$\langle E\rangle \approx 3N\hbar\omega_0\mathrm{e}^{-\beta\hbar\omega_0}$$

$$C = \frac{\partial}{\partial T}\langle E\rangle \approx 3Nk_B(\beta\hbar\omega_0)^2\mathrm{e}^{-\beta\hbar\omega_0}$$

Problem 30

Consider a one-dimensional Ising system with Hamiltonian $H_N = -J(s_1s_2 + s_2s_3 + \cdots + s_{N-1}s_N)$, where $J > 0$, $s_i = \pm 1$ and N is the number of particles (spins) in the system. The canonical partition function is given by: $Z_N = \sum_{\{s_i\}} \mathrm{e}^{-\beta H_N}$. We are in the grand canonical ensemble and the chemical potential is μ. Consider the high temperature limit $k_BT \gg J$. The grand partition function Θ is a convergent series when $N \to \infty$ provided that :

Select one alternative:

- ○ $\mu \lesssim J$
- ○ $\mu \lesssim -k_BT\ln(2)$
- ○ $\mu \lesssim -J$
- ○ $\mu \lesssim -\frac{J^2}{k_BT}$
- ○ $\mu \lesssim k_BT\ln(4)$
- ○ $\mu \lesssim \frac{1}{2}k_BT$

Solution 30 Partition function 1D Ising model:

$$Z = \sum_{s_1=\pm1}\sum_{s_2=\pm1}\cdots\sum_{s_N=\pm1} \mathrm{e}^{\beta J s_1 s_2}\cdots\mathrm{e}^{\beta J s_{N-1}s_N}$$

When $k_BT \gg J$ the partition function is:

$$Z \approx \sum_{s_1=\pm1}\sum_{s_2=\pm1}\cdots\sum_{s_N=\pm1} 1 = 2^N$$

Grand canonical:

$$\Theta = \sum_N Z_N \mathrm{e}^{\beta\mu N} \approx \sum_N 2^N \mathrm{e}^{\beta\mu N} = \sum_N \mathrm{e}^{(\beta\mu - \ln(2))N}$$

Which is a geometric series that converges whenever:

$$\mu < -k_BT\ln(2)$$

Problem 31

Consider a classical ideal gas in two dimensions, confined inside an area A.The grand partition function is:

Select one alternative:

- ○ $\Theta = \exp\left[A\exp(\beta\mu)\right]$
- ○ $\Theta = \exp\left[\frac{h^2}{2\pi m (k_BT)^2} A\exp(-\beta\mu)\right]$
- ○ $\Theta = \exp\left[\frac{h^2}{2\pi m k_BT} A\exp(\beta\mu)\right]$
- ○ $\Theta = \exp\left[\frac{2\pi m (k_BT)^2}{h^4} A\exp(-\beta\mu)\right]$
- ○ $\Theta = \exp\left[\frac{(2\pi m k_BT)^2}{h^2} A\exp(\beta\mu)\right]$
- ○ $\Theta = \exp\left[\frac{2\pi m k_BT}{h^2} A\exp(\beta\mu)\right]$

Solution 31

$$\begin{aligned} Z_N &= \frac{1}{h^{2N}N!}\left(\int \mathrm{d}^2p \int \mathrm{d}^2r\, \mathrm{e}^{-\beta\frac{p^2}{2m}}\right)^N \\ &= \frac{1}{h^{2N}N!}\left(\frac{2\pi m}{\beta}A\right)^N \end{aligned}$$

$$\Theta = \sum_{N=0}^{\infty} Z_N \mathrm{e}^{\beta\mu N} = \exp\left[\frac{2\pi m k_BT}{h^2}A\exp(\beta\mu)\right]$$

Problem 32

Consider a crystalline solid material. The atoms vibrate weakly around their equilibrium positions (harmonic approximation). The atoms also perform collective oscillations with all wavelengths from microscopic to macroscopic scales. Which one of these statements is correct for the statistical mechanics of the solid?

Select one alternative:

- ○ The equipartition theorem implies that there is always roughly as much thermal energy stored in long and short wavelength oscillations.
- ○ The equipartition theorem implies that at high temperatures there is much more energy stored in short wavelength oscillations than long wavelength oscillations
- ○ The Debye theory predicts that the entropy goes to zero at high temperatures.
- ○ The Debye theory predicts that at high temperatures most of the thermal energy is stored in long wavelength oscillations.
- ○ The equipartition theorem implies that the heat capacity goes to zero a high temperatures.
- ○ The Debye theory predicts that collective oscillations are irrelevant at low temperatures.

Solution 32 There are many more short wavelength modes than long wavelength modes. The equipartition theorem implies that at high temperatures there is much more energy stored in short wavelength oscillations than long wavelength oscillations.

Problem 33

Which one of these statements is correct?

Select one alternative:

- ○ The microcanonical ensemble is derived from the canonical ensemble.
- ○ The grand canonical ensemble cannot be derived from the canonical ensemble.
- ○ In the canonical ensemble both the particle number and total energy are constant.
- ○ The grand canonical ensemble can be derived from both the canonical and microcanonical ensemble.
- ○ The pressure in the microcanonical ensemble is zero since the system is isolated.
- ○ The entropy is extensive in the canonical, but cannot be extensive in the microcanonical due to the Gibb's paradox.

Solution 33 The grand canonical ensemble can be derived from both the canonical and microcanonical ensemble.

Problem 34

Consider a particle in one dimension with Hamiltonian: $H = \frac{p_z^2}{2m} + U(z)$. The particle is confined to the interval $0 < z < L$. The potential is $U(z) = \epsilon$ when $0 < z < \ell$, and $U(z) = 0$ when $\ell < z < L$, where ℓ is a constant ($\ell < L$). The pressure at $z = L$ is defined as $P = -\frac{\partial F}{\partial L}$ where F is the free energy. The pressure P is:

Select one alternative:

- $P = \frac{k_B T}{L - \ell + \ell \exp\left(-\frac{\epsilon}{k_B T}\right)}$
- $P = \frac{k_B T}{L + \ell \exp\left(\frac{\epsilon}{k_B T}\right)}$
- $P = \frac{k_B T}{L}$
- $P = \frac{k_B T}{L} \exp\left(-\frac{\ell \epsilon}{L k_B T}\right)$
- $P = \frac{k_B T}{L}\left[L - \ell + \ell \exp\left(-\frac{\epsilon}{k_B T}\right)\right]$
- $P = \frac{k_B T}{L} \exp\left(-\frac{\epsilon}{k_B T}\right)$

Solution 34

$$Z = \frac{1}{h} \int \mathrm{d}z \int \mathrm{d}p_z \, \mathrm{e}^{-\beta H} = \frac{1}{h} \left(\frac{2\pi m}{h^2}\right)^{1/2} \left[\ell \, \exp\left(-\beta\epsilon\right) + L - \ell\right]$$

$$F = -k_B T \ln Z$$

$$P == -\frac{\partial F}{\partial L} = \frac{k_B T}{L - \ell + \ell \exp\left(-\frac{\epsilon}{k_B T}\right)}$$

Problem 35

A system of N particles is confined inside a volume V. The number of states of the system is $\Gamma(N,V,E) = KN^2VE^2$, where K is a constant and E the total energy of the system. We are in the microcanonical ensemble, the entropy is $S = k_B \ln \Gamma$.The relation between energy and temperature is:

Select one alternative:

- ○ $E = 2k_BT$
- ○ $E = Nk_BT$
- ○ $E = \frac{k_BT}{N^2}$
- ○ $E = N^2k_BT$
- ○ $E = \frac{N}{\sqrt{KV}}$
- ○ $E = \frac{N(k_BT)^2}{\sqrt{KV}}$

Solution 35

$$\frac{dS}{dE} = \frac{1}{T} \Rightarrow \frac{2k_B}{E} = \frac{1}{T}$$
$$E = 2k_BT$$

Problem 36

Consider a system with two energy levels, a ground state and one excited state: $\epsilon_0 = 0$, $\epsilon_1 = \epsilon$. The system is populated with N bosons following Bose-Einstein statistics. The canonical partition function is:

Select one alternative:

- ○ $\frac{N(N+1)}{2}\mathrm{e}^{-\beta\epsilon}$
- ○ $N!\,\mathrm{e}^{-\beta\epsilon}$
- ○ $N\mathrm{e}^{-\beta\epsilon}$
- ○ $\frac{1+\mathrm{e}^{-(N-1)\beta\epsilon}}{1+\mathrm{e}^{-\beta\epsilon}}$
- ○ $\mathrm{e}^{\beta\epsilon\mathrm{N}}$
- ○ $\frac{1-\mathrm{e}^{-(N+1)\beta\epsilon}}{1-\mathrm{e}^{-\beta\epsilon}}$

Solution 36 The total energy of the system: $E = n_1\epsilon$, where n_1 is the number of particles in the excited state. Since the particles are indistinguishable there is only one state per energy level and the partition function becomes:

$$Z = \sum_{n_1=0}^{N} \mathrm{e}^{-\beta n_1 \epsilon} = \frac{1-\mathrm{e}^{-(N+1)\beta\epsilon}}{1-\mathrm{e}^{-\beta\epsilon}}$$

Problem 37

A system with four energy levels $\epsilon_0 = 0$, $\epsilon_1 = \epsilon$, $\epsilon_2 = 2\epsilon$ and $\epsilon_3 = 3\epsilon$ is populated by $N = 3$ indistinguishable fermions obeying Fermi-Dirac statistics. The canonical partition function is:

Select one alternative:

- ○ $Z = \mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon} + \mathrm{e}^{-5\beta\epsilon}$
- ○ $Z = \mathrm{e}^{-\beta\epsilon} + \mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon} + \mathrm{e}^{-6\beta\epsilon}$
- ○ $Z = 2\mathrm{e}^{-2\beta\epsilon} + 2\mathrm{e}^{-6\beta\epsilon}$
- ○ $Z = \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon} + \mathrm{e}^{-5\beta\epsilon} + \mathrm{e}^{-6\beta\epsilon}$
- ○ $Z = 1 + \mathrm{e}^{-\beta\epsilon} + \mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-3\beta\epsilon}$

Solution 37 Let (n_0, n_1, n_2, n_3) be the occupation number of the energy levels. Since the particles are fermions there are four different allowed configurations of the system: $(0,1,1,1)$, $(1,0,1,1)$,$(1,1,0,1)$,$(1,1,1,0)$ with corresponding energies 6ϵ, 5ϵ, 4ϵ, 3ϵ. The partition function is therefore:

$$Z = \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon} + \mathrm{e}^{-5\beta\epsilon} + \mathrm{e}^{-6\beta\epsilon}$$

Problem 38

Consider a system of bosons with a finite number of energy levels (M). The energy levels are: $E_n = n\epsilon_0$, where $n = 1, 2, \cdots, M$ and ϵ_0 is a positive constant. We are in the grand canonical ensemble and the chemical potential is $\mu = 0$. The temperature is very high: $k_B T \gg M\epsilon_0$. Which expression is correct for the average energy of the system?

Select one alternative:

- ○ $\langle E \rangle \approx M k_B T$
- ○ $\langle E \rangle \approx M k_B T \mathrm{e}^{-\beta\epsilon_0}$
- ○ $\langle E \rangle \approx k_B T$
- ○ $\langle E \rangle \approx M\epsilon$
- ○ $\langle E \rangle \approx \epsilon$
- ○ $\langle E \rangle \approx k_B T \mathrm{e}^{-M\beta\epsilon_0}$

Solution 38

$$\langle E \rangle = \sum_{n=1}^{M} \frac{E_n}{\mathrm{e}^{\beta(E_n - \mu)} - 1} \approx \sum_{n=1}^{M} \frac{E_n}{1 + \beta(E_n - \mu) - 1}$$

When $\mu = 0$ we therefore get:

$$\langle E \rangle \approx \sum_{n=1}^{M} \frac{1}{\beta} = M k_B T$$

Problem 39

A system has three energy levels, a ground state $\epsilon_1 = 0$ and two excited states with the same energy $\epsilon_2 = \epsilon$, $\epsilon_3 = \epsilon$. The system is populated with bosons obeying Bose-Einstein statistics. We are in the grand canonical ensemble. The average number of particles in the system is $\langle N \rangle = 3$. Whe are in the high temperature regime $k_B T \gg \epsilon$. Which expression is correct for the chemical potential:

Select one alternative:

- ○ $\mu \approx -3\epsilon_0$
- ○ $\mu \approx -k_B T \ln(2)$
- ○ $\mu \approx -\epsilon_0 \ln(2)$
- ○ $\mu \approx -3 k_B T$
- ○ $\mu \approx -3\frac{(k_B T)^2}{\epsilon_0}$
- ○ $\mu \approx -3\frac{\epsilon_0}{(k_B T)^2}$

Solution 39

$$\langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle = \frac{1}{\mathrm{e}^{-\beta\mu} - 1} + \frac{2}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1} \approx \frac{3}{\mathrm{e}^{-\beta\mu} - 1}$$

since $\mathrm{e}^{\beta\epsilon} \approx 1$. Since $\langle n_1 + n_2 + n_3 \rangle = 3$ we must have:

$$\mu \approx -k_B T \ln(2)$$

Problem 40

Consider a classical harmonic oscillator $H = \frac{p_x^2}{2m} + \frac{K}{2}x^2$. We are in the canonical ensemble. What is the fluctuation in energy $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$?

Select one alternative:

- ○ $\Delta E^2 = \frac{(k_B T)^3}{K}$
- ○ $\Delta E^2 = k_B T\sqrt{\frac{K}{m}}$
- ○ $\Delta E^2 = (k_B T)^2$
- ○ $\Delta E^2 = \frac{(k_B T)^3}{\sqrt{Km}}$
- ○ $\Delta E^2 = 2(k_B T)^2$
- ○ $\Delta E^2 = 0$

Solution 40

Equipartition theorem:

$$\langle E\rangle = \frac{1}{2}k_BT + \frac{1}{2}k_BT = k_BT$$

Fluctuation in energy:

$$\Delta E^2 = -\frac{\partial}{\partial\beta}\langle E\rangle = \frac{1}{\beta^2} = (k_BT)^2$$