NTNU **Institutt for fysikk**

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Exam in TFY4230 STATISTICAL PHYSICS

Tuesday august 9th, 2011
09:00–13:00

Allowed help: Alternativ **C**

Standard calculator (according to list prepared by NTNU).
K. Rottman: *Matematisk formelsamling* (all languages).
Barnett & Cronin: *Mathematical Formulae*

Det er også en norsk versjon av dette eksamenssettet.

This problemset consists of 2 pages.

Problem 1. Chains of Ising spins

A cyclic chain of three Ising spins have the Hamiltonian

$$H = J\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right). \tag{1}$$

a) Write down all configurations and the corresponding energies for this chain.

b) Find the partition function $Z = \mathrm{e}^{-\beta F} = \mathrm{e}^{-\beta U + S/k_B}$ for this chain.

c) Find the mean energy $U = \langle H \rangle$ for this chain. Consider in particular the limit $T \to 0$, both for $J > 0$ and $J < 0$.

d) Find the heat capacity $C = \frac{\partial U}{\partial T}$ of this chain.

e) Find the entropy S of this chain. Consider in particular the limit $T \to 0$, both for $J > 0$ and $J < 0$.

f) Find the correlation function $\langle s_1 s_2 \rangle$ for this chain.

g) Sketch how you would do the correponding analysis for a cyclic chain with N Ising spins, i.e. with Hamiltonian $H = J\left(s_N s_1 + \sum_{j=1}^{N-1} s_j s_{j+1}\right)$.

Problem 2. Quantum magnetism

The one-particle Hamiltonian for an electron (charge $q = -e$) in a magnetic field $\boldsymbol{B}$ is

$$H = \frac{1}{2m_e}\left(\boldsymbol{p} + e\boldsymbol{A}\right)^2 - g\mu_B B s_z, \tag{2}$$

where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. After quantization the eigenenergies of H are found to be

$$E = \frac{1}{2m_e}p_z^2 + \left(n + \frac{1}{2}\right)\varepsilon_a + s_z\,\varepsilon_b, \quad \text{where } s_z = \pm\frac{1}{2} \text{ and } n = 0, 1, \ldots. \tag{3}$$

Here $\varepsilon_a = \mu_B B$ and $\varepsilon_b = \frac{1}{2}g\mu_B B$. For each value of p_z, n, and s_z there are $eB\mathcal{A}/h$ degenerate states, where $\mathcal{A}$ is the area of the system normal to the magnetic field.
In the grand canonical ensemble the partition function for a gas of such electrons can be expressed as

$$\beta p = \frac{\ln \Xi}{V} = \frac{eB\sqrt{2m_e}}{h^2}\sum_{s_z=\pm\frac{1}{2}}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\mathrm{d}\varepsilon_z}{\sqrt{\varepsilon_z}}\,\ln\left\{1 + \mathrm{e}^{-\beta[\varepsilon_z + (n+1/2)\varepsilon_a + s_z\varepsilon_b - \mu]}\right\}, \tag{4}$$

when we ignore the interaction between electrons.

a) Sketch the general connection between one-particle states E_n and the grand canonical partition function Ξ for an ideal Fermi gas.

b) Indicate how one in this particular case arrive at the partition function (4) from the one-particle states (3).

c) Show that the partition function (4) can be rewritten in the form

$$\beta p = \sum_{M=1}^{\infty}\frac{(-1)^{M+1}}{M}\,\mathrm{e}^{M\beta\mu} \times \frac{eB\sqrt{2m_e}}{h^2} \times$$
$$\times \sum_{s_z=\pm 1/2}\mathrm{e}^{-s_z M\beta\varepsilon_b}\sum_{n=0}^{\infty}\mathrm{e}^{-(n+1/2)M\beta\varepsilon_a}\int_0^{\infty}\frac{\mathrm{d}\varepsilon_z}{\sqrt{\varepsilon_z}}\,\mathrm{e}^{-M\beta\varepsilon_z}. \tag{5}$$

d) Perform the summations over s_z and n, and the integration over ε_z in equation (5).

e) Consider the limit $B \to 0$ of your result from point **d)**. Does the result look like the partition function for an ideal electron gas?

f) The average magnetization per volume unit is here given by the expression

$$M = \left(\frac{\partial p}{\partial B}\right)_{\beta,\mu}. \tag{6}$$

Calculate this expression to first order in the fugacity $z = \lambda^{-3}\,\mathrm{e}^{\beta\mu}$, where $\lambda = h^2/\sqrt{2\pi k_B T m_e}$ is the thermal *de Broglie* wavelength of the electron. You may assume that the quantity $u \equiv \beta\mu_B B$ is small, and only calculate M to first order in u.

g) For which values of g is the system *paramagnetic*, and for which values of g is it *diamagnetic*?

Given: Some of the expressions below *may* be of use for solving this exam set.

$$(1-x)^{-1} = \sum_{M=0}^{\infty} x^M, \tag{7}$$

$$\ln(1+x) = \sum_{M=1}^{\infty}\frac{(-1)^{M+1}}{M}\,x^M, \tag{8}$$

$$\int_0^{\infty}\frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{e}^{-t} = \sqrt{\pi}. \tag{9}$$