Exam TFY4230 Statistical Physics kl 09.00 - 13.00 Wednesday 01. June 2016

Problem 1. **Ising ring (Points: 10+10+10 = 30)**
A system of Ising spins $\sigma_i = \pm 1$ on a ring with periodic boundary conditions is defined by the Hamiltonian

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i$$

where i denotes a lattice site, and $\sigma_{N+1} = \sigma_1$. J is the strength of the nearest neighbor interaction between spins, and h is a uniform external magnetic field. The partition function for this system is given by

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H} = e^{-\beta G},$$

where G is the Gibbs energy of the system. An explicit calculation yields $Z = \lambda_+^N + \lambda_-^N$, where

$$\lambda_\pm = e^K \left[\cosh(\omega) \pm \sqrt{\sinh^2(\omega) + e^{-4K}} \right],$$

where $K \equiv \beta J$ and $\omega \equiv \beta h$. Here, $\beta \equiv 1/k_B T$, k_B is Boltzmann's constant, and T is temperature.

a.
Write out the sum in the partition function explicitly for $N = 3$, collecting all terms of equal Boltzmann weight $e^{-\beta H}$.
Show that the magnetization $m \equiv \lim_{N\to\infty}(M/N) = \lim_{N\to\infty}(1/N)\sum_{i=1}^{N}\langle\sigma_i\rangle$ of this system is given by the expression

$$m = \frac{\sinh(\omega)}{\sqrt{\sinh^2(\omega) + e^{-4K}}}.$$

b.
Explain on physical grounds the difference in the results for $J > 0$ and $J < 0$.

c.

Consider now a slightly different model of Ising-spins on a ring with the following Hamiltonian

$$H = -\sum_{i=1}^{N} [J_1 \; \sigma_i \sigma_{i+1} + J_2 \; \sigma_i \sigma_{i+2}]$$

where $J_1 > 0$ is the interaction strength between nearest neighbor spins, and $J_2 > 0$ is the interaction strength between next-nearest neighbor spins. There is no external magnetic field. Compute the Gibbs energy G for this system for $N \to \infty$. From this, find the limiting values of G for low temperatures $\beta J_1 \gg 1$, $\beta J_2 \gg 1$. Give a physical explanation of the result.
(Hint: Introduce the new spin variable $\tau_i = \sigma_i \sigma_{i+1}$ and use periodic boundary conditions on the τ_i.)

Problem 2. Ideal gas in a $3D$ anharmonic trap (Points: 10+10+10=30)

The canonical partition function Z for a system of N classical non-relativistic particles of equal mass m which are in thermal equilibrium with their surroundings and moving in three spatial dimension $3D$ in an anharmonic trap potential, is given by

$$Z = \frac{1}{N!h^{3N}} \int d\mathbf{r}_1..d\mathbf{r}_N \int d\mathbf{p}_1..d\mathbf{p}_N \; e^{-\beta H} = e^{-\beta F}$$

where $\beta = 1/k_B T$, k_B is Boltzmann's constant, T is temperature, h is Plank's constant, $F = U - TS$ is the Helmholz free energy, and the Hamiltonian H of the system is given by

$$\begin{aligned} H &= \sum_{i=1}^{N} H_i \\ H_i &= \frac{\mathbf{p}_i^2}{2m} + \alpha |\mathbf{r}_i|^3. \end{aligned}$$

Here, α is a dimensionful constant which gives the strength of the anharmonic trap-potential $\alpha|\mathbf{r}_i|^3$. The $3D$ volume of the system to which the particles are confined is defined by a sphere of radius R, with volume $V = 4\pi R^3/3$. The coordinates $\{\mathbf{r}_i\}$ are all measured from the center of this sphere.

a.
Show that the partition function of the system is given by

$$\begin{aligned} Z &= \frac{1}{N!} \frac{V^N}{\Lambda^{3N}} \left[\frac{1 - e^{-x}}{x} \right]^N \\ x &\equiv \frac{3\beta\alpha V}{4\pi} \\ \Lambda &\equiv \frac{h}{\sqrt{2\pi m k_B T}}. \end{aligned}$$

b.
Compute the internal energy $U = \langle H \rangle$ of the system for the limits $3\beta\alpha V/4\pi \ll 1$ and $3\beta\alpha V/4\pi \gg 1$. Explain your results on physical grounds in each case.

c.
Compute the pressure of the system for general values of $3\beta\alpha V/4\pi$. Consider then the limits $3\beta\alpha V/4\pi \ll 1$ and $3\beta\alpha V/4\pi \gg 1$ and compute the equation of state in these limits. Explain your results on physical grounds in each case. (Hint: Use $F = U - TS$ and $TdS = dU + pdV$ to express pressure p in terms of Z.)
Useful formulae:

$$\begin{aligned} U &= \frac{1}{Z} \frac{1}{N!h^{3N}} \int d\mathbf{r}_1..d\mathbf{r}_N \int d\mathbf{p}_1..d\mathbf{p}_N \;\; H \;\; e^{-\beta H} \\ \int d^\nu \mathbf{r} \; F(|\mathbf{r}|) &= \Omega_\nu \int dr \; r^{\nu-1} \; F(r); \quad \Omega_\nu = \frac{2\pi^{d/2}}{\Gamma(d/2)}; \quad \Gamma(z+1) = z\,\Gamma(z) \\ \int_0^a dx \; x^{\nu-1} \; e^{-x^\nu} &= \frac{1}{\nu} \int_0^{a^\nu} du \; e^{-u} \end{aligned}$$

Problem 3. d**-dimensional Bose system (Points: 10+10+10=30)**

Ultra-relativistic non-interacting bosons moving in d dimensions have a Hamiltonian given by

$$\begin{aligned} H &= \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\, n_{\mathbf{k}}; \quad n_{\mathbf{k}} = 0, 1, 2, ... \\ \varepsilon_{\mathbf{k}} &= \hbar c|\mathbf{k}| \end{aligned}$$

where c is the speed of light, $\hbar = h/2\pi$ with h Planck's constant, and $\mathbf{k}$ is a wavenumber uniquely determining the single-particle bosonic states.
The grand canonical partition function Z_g for a system of non-interacting bosons is given by

$$\ln Z_g = -\sum_{\mathbf{k}} \ln\left[1 - e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}\right] = \beta p V$$

Here, $\beta = 1/k_B T$, k_B is Boltzmann's constant, T is temperature, p is pressure and V is the volume of the system. It can be shown (need not be shown!) that the density of states $g(e)$ for this system is given by

$$g(e) = \frac{V}{(2\pi\hbar c)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)}\, e^{d-1}\Theta(e)$$

where $\Theta(e) = 1, e \geq 0; \Theta(e) = 0, e < 0$.

a.
The average number of particles in the system is given by $\langle N \rangle = \partial \ln Z_g/\partial(\beta\mu) = \sum_{\mathbf{k}} n_{\mathbf{k}}$ where $n_{\mathbf{k}}$ is the average number of particles with wavenumber $\mathbf{k}$. The internal energy U is given by $U = \sum_{\mathbf{k}}\ \varepsilon_{\mathbf{k}}\ n_{\mathbf{k}}$. Show that

$$\begin{aligned} \langle N \rangle &= \sum_{\mathbf{k}} \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} - 1} \\ U &= \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} - 1} \end{aligned}$$

b.
Introduce the density of states $g(e)$ and the fugacity $z = e^{\beta\mu}$. Show that

$$\begin{aligned} \langle N \rangle &= V \sum_{l=1}^{\infty} l\ b_l\ z^l \\ \frac{\beta U}{d} &= V \sum_{l=1}^{\infty} b_l\ z^l \end{aligned}$$

and thereby determine b_l. Compute the ratio U/pV.

c.
Compute the first and second virial coefficients $E_1(T)$ and $E_2(T)$ in the virial expansion for the internal energy, $U = E_1(T)\rho + E_2(T)\rho^2 + ...$ Show that $E_2(T) \neq 0$ is a quantum effect, and give a physical explanation for the sign of $E_2(T)$.

Useful formulae:

$$
\begin{aligned}
\sum_{\mathbf{k}} F(\varepsilon_{\mathbf{k}}) &= \int_{-\infty}^{\infty} de\ g(e)\ F(e) \\
g(e) &\equiv \sum_{\mathbf{k}} \delta(e - \varepsilon_{\mathbf{k}}) \\
\Gamma(z) &= \int_0^{\infty} dx\ x^{z-1}\ e^{-x} \\
\Gamma(z+1) &= z\ \Gamma(z)
\end{aligned}
$$