Exam TFY4230 Statistical Physics kl 09.00 - 13.00 Wednesday 10. August 2016

Problem 1. Ising ring (Points: 10+10+10 = 30)
A system of Ising spins $\sigma_i = \pm 1$ on a ring with periodic boundary conditions is defined by the Hamiltonian

$$H = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i$$

where i denotes a lattice site, and $\sigma_{N+1} = \sigma_1$. J is the strength of the nearest neighbor interaction between spins, and h is a uniform external magnetic field. The partition function for this system is given by

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H} = e^{-\beta G},$$

where G is the Gibbs energy of the system. An explicit calculation yields $Z = \lambda_+^N + \lambda_-^N$, where

$$\lambda_\pm = e^K\left[\cosh(\omega) \pm \sqrt{\sinh^2(\omega) + e^{-4K}}\right],$$

where $K \equiv \beta J$ and $\omega \equiv \beta h$. Here, $\beta \equiv 1/k_B T$, k_B is Boltzmann's constant, and T is temperature.

a.
Show that in general, for such a spin system, the magnetization is given by

$$M \equiv \langle\sum_{i=1}^{N}\sigma_i\rangle = k_B T\left(\frac{\partial \ln Z}{\partial h}\right)_T$$

b.
From this, find an expression for the magnetization $m \equiv (M/N) = (1/N)\sum_{i=1}^{N}\langle\sigma_i\rangle$ of this system for general N, and show that for very large N, it is given by the expression

$$m = \frac{\sinh(\omega)}{\sqrt{\sinh^2(\omega) + e^{-4K}}}.$$

c.

Consider now a slightly different model of Ising-spins on a ring with the following Hamiltonian

$$H = -\sum_{i=1}^{N} \left[J_1 \ \sigma_i \sigma_{i+1} + J_2 \ \sigma_i \sigma_{i+2} \right]$$

where $J_1 > 0$ is the interaction strength between nearest neighbor spins, and $J_2 > 0$ is the interaction strength between next-nearest neighbor spins. There is no external magnetic field.
This model may be re-expressed in terms of new Ising variables $\tau_i = \sigma_i \sigma_{i+1}$, with periodic boundary conditions such that $\tau_{N+1} = \tau_1$.
Compute the expectation values $\langle \sigma_i \sigma_{i+1} \rangle$ and $\langle \sigma_i \sigma_{i+2} \rangle$ in the limit $N \to \infty$ for general values of β. Explain on physical grounds the results for $\beta \to 0$ and $\beta \to \infty$.

Problem 2. Ideal gas in a dD anharmonic trap (Points: 10+10+10=30)

The canonical partition function Z for a system of N classical non-relativistic particles of equal mass m which are in thermal equilibrium with their surroundings and moving in d spatial dimension dD in an anharmonic trap potential, is given by

$$Z = \frac{1}{N!h^{dN}} \int d\mathbf{r}_1..d\mathbf{r}_N \int d\mathbf{p}_1..d\mathbf{p}_N \;\; e^{-\beta H}$$

where $\beta = 1/k_BT$, k_B is Boltzmann's constant, T is temperature, h is Plank's constant, $F = U - TS$ is the Helmholz free energy, U is the internal energy, S is the entropy, and the Hamiltonian H of the system is given by

$$H = \sum_{i=1}^{N} H_i$$
$$H_i = \frac{\mathbf{p}_i^2}{2m} + \alpha |\mathbf{r}_i|^d.$$

Here, α is a dimensionful constant which gives the strength of the anharmonic trap-potential $\alpha|\mathbf{r}_i|^d$. The dD volume of the system to which the particles are confined is defined by a sphere of radius R, with volume $V = \Omega_d R^d/d$. Here, Ω_d is the solid angle in d dimensions. The coordinates $\{\mathbf{r}_i\}$ are all measured from the center of this sphere.

a.
Show that the partition function of the system is given by

$$Z = \frac{1}{N!}\frac{V^N}{\Lambda^{dN}}\left[\frac{1-e^{-x}}{x}\right]^N$$
$$x \equiv \frac{d\beta\alpha V}{\Omega_d}$$
$$\Lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}.$$

b.
Compute the internal energy $U = \langle H \rangle$ and the entropy S of the system.
Useful formulae:

$$U = \frac{1}{Z}\frac{1}{N!h^{dN}} \int d\mathbf{r}_1..d\mathbf{r}_N \int d\mathbf{p}_1..d\mathbf{p}_N \;\; H \quad e^{-\beta H}$$
$$\int d^\nu \mathbf{r}\; F(|\mathbf{r}|) = \Omega_\nu \int dr\; r^{\nu-1}\; F(r); \quad \Omega_\nu = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}; \quad \Gamma(z+1) = z\,\Gamma(z)$$
$$\int_0^a dx\; x^{\nu-1}\; e^{-x^\nu} = \frac{1}{\nu}\int_0^{a^\nu} du\; e^{-u}$$

Problem 3. 2-dimensional Fermi system (Points: 10+10+10=30)

Non-interacting ultra-relativistic spin-1/2 fermions moving in 2 dimensions have a Hamiltonian given by

$$\begin{aligned} H &= \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}}\, n_{\mathbf{k}\sigma}; \quad n_{\mathbf{k}} = 0, 1; \sigma = \pm 1. \\ \varepsilon_{\mathbf{k}} &= \hbar c |\mathbf{k} \end{aligned}$$

where c is the speed of light, $\hbar = h/2\pi$ with h Planck's constant, and $\mathbf{k}$ is a wavenumber uniquely determining the single-particle states.
The grand canonical partition function Z_g for a system of non-interacting fermions is given by

$$\ln Z_g = \sum_{\mathbf{k},\sigma} \ln \left[1 + e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}\right] = \beta p V$$

Here, $\beta = 1/k_B T$, k_B is Boltzmann's constant, T is temperature, p is pressure and V is the volume of the system. The density of states $g(\varepsilon)$ per spin for this system is given by

$$g(\varepsilon) = \frac{V}{(2\pi\hbar c)^2}\, 2\pi\, \varepsilon \Theta(\varepsilon)$$

where $\Theta(x) = 1, x \geq 0; \Theta(x) = 0, x < 0$.
The average number of particles in the system is given by $\langle N \rangle = \langle \sum_\sigma N_\sigma \rangle = \partial \ln Z_g / \partial(\beta\mu) = \sum_{\mathbf{k}\sigma} n_{\mathbf{k}\sigma}$ where $n_{\mathbf{k}\sigma}$ is the average number of particles with wavenumber $\mathbf{k}$ and spin σ. The internal energy U is given by $U = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}}\, n_{\mathbf{k}\sigma}$. Introduce the density of states $g(\varepsilon)$ and the fugacity $z = e^{\beta\mu}$.

a.
Show that

$$\begin{aligned} \langle N \rangle &= V \sum_{l=1}^{\infty} l\, b_l\, z^l \\ \frac{\beta U}{2} &= V \sum_{l=1}^{\infty} b_l\, z^l \end{aligned}$$

and thereby determine b_l. Compute the ratio U/pV.

b.
Compute the pressure at $T = 0$. What is the classical limit of this result?

c.
Compute the magnetization m of this system, where $m = (1/N)(\langle N_{\sigma=1} \rangle - \langle N_{\sigma=-1} \rangle)$.

Useful formulae:

$$\begin{aligned}
\sum_{\mathbf{k}} F(\varepsilon_{\mathbf{k}}) &= \int_{-\infty}^{\infty} d\varepsilon \; g(\varepsilon) \; F(\varepsilon) \\
g(\varepsilon) &\equiv \sum_{\mathbf{k}} \delta(\varepsilon - \varepsilon_{\mathbf{k}}) \\
\Gamma(z) &= \int_{0}^{\infty} dx \; x^{z-1} \; e^{-x} \\
\Gamma(z+1) &= z \, \Gamma(z)
\end{aligned}$$