Problem 1. **(Points: 10+10+10= 30)**

A classical non-relativistic system of particles of mass m defined within a three-dimensional volume V in the grand canonical ensemble, has a partition function given by

$$\begin{aligned} Z_g &= e^{\beta pV} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N \\ Z_N &\equiv \frac{1}{N! h^{3N}} \int d\Gamma \; e^{-\beta H_N} \end{aligned}$$

Here, $d\Gamma = \prod_{i=1}^{N} d^3 r_i d^3 p_i$ where $\{r_i, p_i\}$ are the coordinates and momenta of N particles. p is the pressure of the system, $\beta = 1/k_B T$, μ is the chemical potential, and N is the total number of particles contained within the volume V. N is allowed to fluctuate around some average value $\langle N \rangle$. k_B is Boltzmann's constant, T is temperature, and h is Planck's constant with dimension Js (Joulesecond). H_N is the Hamiltonian of such as system with N particles, and is given by

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V_p(\mathbf{r}_1, ..., \mathbf{r}_N)$$

where V_p is the potential energy of the system. We define the configuration integral as follows

$$Q_N \equiv \int .. \int d^3 r_1 ... d^3 r_N e^{-\beta V_p(\mathbf{r}_1, ..., \mathbf{r}_N)}$$

with the definition $Q_0 \equiv 1$. The integrations over the coordinates are carried out over the volume V of the system.

a) Carry out the momentum integrations in Z_g, and show that Z_g may be written as

$$Z_g = \sum_{N=0}^{\infty} \frac{z^N}{N!} \; Q_N$$

and thus give an expression for z.

b) Introducing the density $\rho = \langle N \rangle / V$, where $\langle N \rangle$ is the average number of particles in the system, show that we may write

$$\begin{aligned} \beta p &= A_1 z + A_2 z^2 + ... \\ \rho &= A_1 z + 2A_2 z^2 + ... \end{aligned}$$

thereby expressing (A_1, A_2) in terms of Q_1 and Q_2.

c) We now specialize to the case where $V_p(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{i=1}^{N} \phi(\mathbf{r}_i)$, where ϕ is an analytic function of its arguments, bounded from below, but otherwise arbitrary. Compute the virial coefficients $B_l(T); l \geq 2$ for the system. Give a physical interpretation of the result.

<u>Problem 2.</u> (Points: 5+10+10+5=30)

A dilute system of N magnetic impurities may be described as a system of independent spins $\{\mathbf{S}_i\}, i \in (1,..N)$. In a uniform external magnetic field $\mathbf{B}$, the Hamiltonian for such a system is given by

$$H = -\mathbf{B} \cdot \sum_{i=1}^{N} \mathbf{S}_i$$

We consider a system where $\mathbf{B} = B\hat{z}$, and $\mathbf{S}_i = \sigma_i \hat{z}$, where $\sigma_i \in (-m, -m+1, ..., m-1, m)$ and m is a positive integral or half-integral number.

a) Compute the partition function

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H}$$

of this system. Here, $\beta = 1/k_B T$, with k_B Boltzmann's constant and T temperature.

b) Compute $\langle H \rangle$ and $M = \langle \sum_i \sigma_i \rangle$. Also, give explicit expressions for $\langle H \rangle$ and M when $\beta B \ll 1$ and $\beta B \gg 1$.

c) Compute the uniform isothermal magnetic susceptibility

$$\chi = \left(\frac{\partial M}{\partial B} \right)_T$$

d) Give a *simple* physical interpretation of the result you find for Z in the limit $T \to \infty$. Explain on physical grounds the behavior of χ in the limits $T \to \infty$ and $T \to 0$.

Problem 3. **(Points: 10+10+10+10=40)**

A three-dimensional gas of non-interacting bosons obeys the dispersion relation $\varepsilon_{\mathbf{k}} = A|\mathbf{k}|^{1/2}$, where $\mathbf{k}$ is a wavenumber and A is a dimensionful constant with dimension $[A] = Jm^{1/2}$. The system is contained in a macroscopically large volume V. The grand canonical partition function of the system is given by

$$Z_g = \prod_{\mathbf{k}} \left(\frac{1}{1 - ze^{-\beta\varepsilon_{\mathbf{k}}}} \right) = e^{\beta p V}$$

where $z = e^{\beta\mu}$. Here, μ is a chemical potential and $\beta = 1/k_B T$, where k_B is Boltzmann's constant and T is temperature. p is the pressure of the system.

a) Find a general expression for the average number of particles in the system, $\langle N \rangle$, expressed as a sum over $\mathbf{k}$. Split off the $\mathbf{k} = 0$-contribution in the sum and call this N_0. Find an expression for N_0 in terms of z.

b) Suppose now that N_0 is a macroscopically large number $N_0 \gg 1$. Compute the value of μ in this case.

c) Introduce the density of particles in the ground state, $n_0 = N_0/V$. Furthermore, define the total density of the system $n = \langle N \rangle / V$. Find an expression for the temperature T_c at which the ground state starts to be macroscopically occupied. Compute n_0/n when $T = T_c/2$.

d) Show by an exact calculation to all orders in z that for all temperatures, the pressure in the ideal Bose-gas defined above is lower than the pressure in the ideal classical gas, at a given temperature T and density n. In particular, compute the ratio of the pressure in the above ideal Bose-gas and the ideal classical gas at $T = T_c$.