Oppgåve 1. Løysing.

a) Av symmetrigrunnar er

$$\langle v_x \rangle = 0$$

b) Ekvipartisjonsprinsippet gir

$$\langle \tfrac{1}{2} m v_x^2 \rangle = \tfrac{1}{2} kT$$

$$\Rightarrow \langle v_x^2 \rangle = \frac{kT}{m}$$

c) Av symmetrigrunnar er

$$\langle v^2 v_x \rangle = 0$$

d) Av symmetrigrunnar er

$$\langle v_x^2 v_y \rangle = 0$$

e) Vi har:

$$\langle (v_x + b v_y)^2 \rangle = \langle v_x^2 \rangle + b^2 \langle v_y^2 \rangle + 2b \langle v_x v_y \rangle$$

$$= \frac{kT}{m} + b^2 \frac{kT}{m} + 0 = (1+b^2)\frac{kT}{m}$$

Oppgåve 2. Løysing

a)

Sannsynlegheten for at systemet skal vera i tilstanden n, er

$$p_n = \frac{e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = \frac{e^{-\beta E_n}}{Z}$$

Middelenergien til systemet er då

$$\underline{\underline{U}} = \langle H \rangle = \sum_n p_n E_n = \frac{1}{Z} \sum_n E_n e^{-\beta E_n}$$

$$= -\frac{1}{Z} \frac{\partial}{\partial \beta} \sum_n e^{-\beta E_n} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = \underline{\underline{-\frac{\partial \ln Z}{\partial \beta}}}$$

b)

Vi har:

$$\underline{\underline{\langle H^2 \rangle}} = \sum_n E_n^2 p_n = \frac{1}{Z} \sum_n E_n^2 e^{-\beta E_n}$$

$$= \frac{1}{Z} \frac{\partial^2}{\partial \beta^2} \left(\sum_n e^{-\beta E_n} \right) = \underline{\underline{\frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}}}$$

c)

Av a) og b) følgjer

$$\langle H^2 \rangle - \langle H \rangle^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \left(-\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 = \frac{\partial^2}{\partial \beta^2}\ln Z$$

Altså

$$\underline{\underline{\Delta U}} = \underline{\underline{\sqrt{\frac{\partial^2}{\partial \beta^2}\ln Z}}}$$

Vi har

$$\frac{\partial^2 \ln Z}{\partial \beta^2} = -\frac{\partial}{\partial \beta}\left(-\frac{\partial}{\partial \beta}\ln Z\right)$$

Vha. a) ser vi da:

$$\underline{\underline{\Delta U}} = \underline{\underline{\sqrt{-\frac{\partial U}{\partial \beta}}}} = \underline{\underline{T\sqrt{k\frac{\partial U}{\partial T}}}}$$

d) Varmekapasiteten

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

Altså blir

$$\underline{\underline{\Delta U}} = \underline{\underline{T\sqrt{k C_V}}}$$

e)

For ein ideell monoatomisk gass er i følgje ekvipartisjonsprinsippet

$$U = \frac{3}{2} N k T$$

Dette gir

$$\Delta U = T\sqrt{k\,\frac{\partial U}{\partial T}} = T\sqrt{k\,\frac{3}{2}Nk}$$

$$= kT\sqrt{\frac{3}{2}N}$$

Det følgjer

$$\underline{\underline{\frac{\Delta U}{U}}} = \frac{kT\sqrt{\frac{3}{2}N}}{\frac{3}{2}NkT} = \underline{\underline{\frac{1}{\sqrt{\frac{3}{2}N}}}}$$

Oppgave 3. Løysing

a)

i) For fermionar er $n_k = 0$ eller 1.

$$\Rightarrow \underline{\underline{\langle n_k \rangle}} = \sum_{n_k=0}^{1} p(n_k) \cdot n_k = p(1)$$

$$= \frac{e^{\beta(\mu-\varepsilon_k)}}{1+e^{\beta(\mu-\varepsilon_k)}} = \underline{\underline{\frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}}} \qquad (1)$$

ii) For bosonar er $n_k = 0, 1, 2, \ldots$

$$\Rightarrow \sum_{n_k=0}^{\infty} e^{\beta(\mu-\varepsilon_k)n_k} = \sum_{n_k=0}^{\infty} \left(e^{\beta(\mu-\varepsilon_k)}\right)^{n_k}$$

$$= \frac{1}{1-e^{\beta(\mu-\varepsilon_k)}}$$

Då blir

$$p(n_k) = \left(1-e^{\beta(\mu-\varepsilon_k)}\right) e^{\beta(\mu-\varepsilon_k)n_k}$$

og

$$\langle n_k \rangle = \sum_{n_k=0}^{\infty} n_k\, p(n_k) = (1-e^{u}) \sum_{n_k=0}^{\infty} n_k e^{u n_k},$$

der $u = \beta(\mu - \varepsilon_k)$.

Vi kan skrive

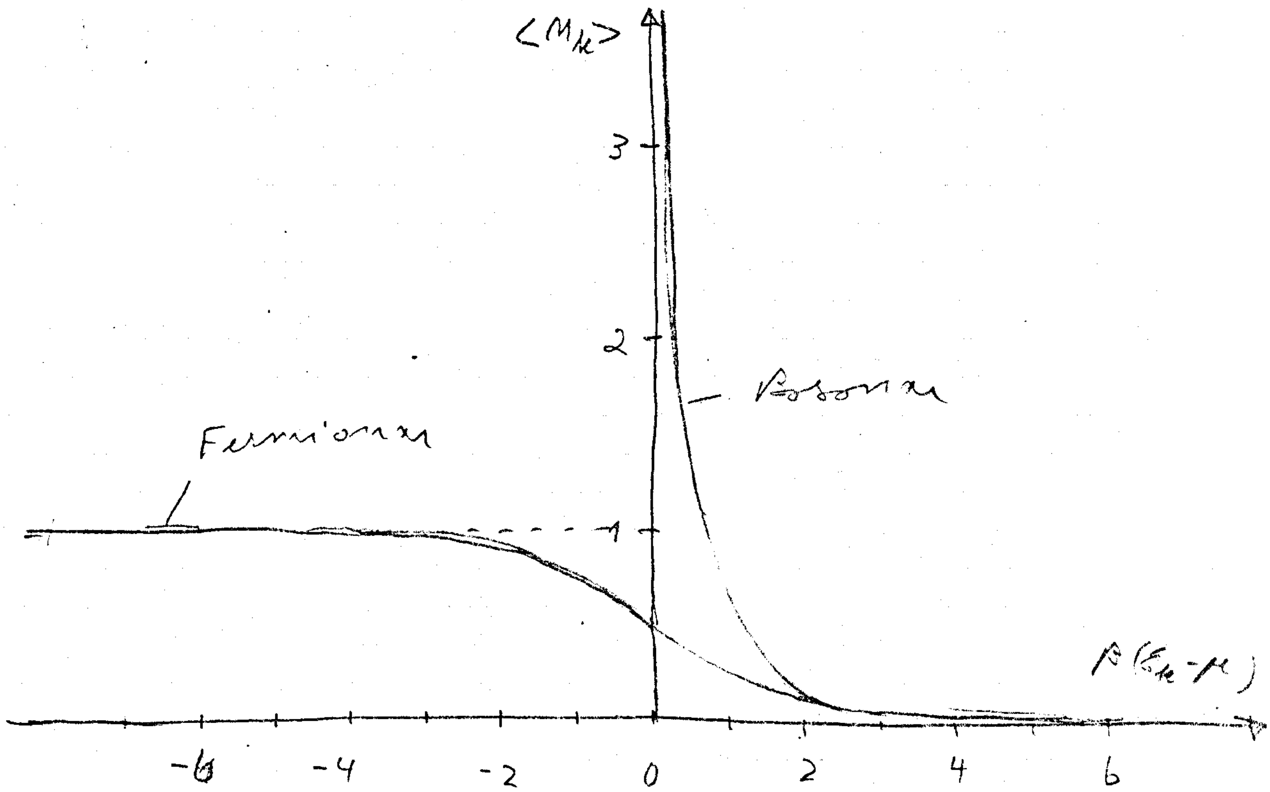
$$\langle n_k \rangle = (1-e^{u}) \frac{\partial}{\partial u} \sum e^{u n_k}$$

$$= (1-e^{u}) \frac{\partial}{\partial u} \frac{1}{1-e^{u}} = \frac{e^{u}}{1-e^{u}}$$

Altså

$$\underline{\underline{\langle n_k \rangle}} = \frac{e^{\beta(\mu-\varepsilon_k)}}{1-e^{\beta(\mu-\varepsilon_k)}} = \underline{\underline{\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}}} \qquad (2)$$

b)



NB! For bosoner må $\varepsilon_k - \mu > 0$. I motsatt fall får en negative besetningstall, noe som er meiningsløst

c) ..

Vi har:

$$\langle n_k^2 \rangle = \sum_{n_k} n_k^2 \, p(n_k)$$

i) Fermioner

$$\langle n_k^2 \rangle = \sum_{n_k=0}^{1} n_k^2 \, p(n_k) = p(1) \overset{(1)}{=} \langle n_k \rangle$$

Altså er

$$\underline{\underline{\Delta n_k}} = \sqrt{\langle n_k^2 \rangle - \langle n_k \rangle^2} = \underline{\underline{\sqrt{\langle n_k \rangle (1 - \langle n_k \rangle)}}}$$

ii) Bosoner

$$\langle n_k^2 \rangle = \sum_{n_k=0}^{\infty} n_k^2 \, p(n_k) = (1 - e^{\mu}) \sum_{n_k=0}^{\infty} n_k^2 e^{\mu n_k} \quad (3$$

Vi har:

$$S = \sum_{n_k=0}^{\infty} e^{\mu n_k} = \frac{1}{1 - e^{\mu}}$$

$$\Rightarrow \quad \sum_{n_k=0}^{\infty} n_k^2 e^{\mu n_k} = \frac{\partial^2 S}{\partial \mu^2} = \frac{\partial}{\partial \mu} \frac{e^{\mu}}{(1 - e^{\mu})^2}$$

$$= \frac{e^{\mu}}{(1-e^{\mu})^2} + 2\frac{(e^{\mu})^2}{(1-e^{\mu})^3}$$

Insertt i (3), gti dalte :

$$\langle n_k^2 \rangle = \frac{e^{\mu}}{1-e^{\mu}} + 2\left(\frac{e^{\mu}}{1-e^{\mu}}\right)^2$$

Në u i përgjigje pikt. a)

$$\langle n_k \rangle = \frac{e^{\mu}}{1-e^{\mu}}$$

Atëherë

$$\langle n_k^2 \rangle = \langle n_k \rangle + 2\langle n_k \rangle^2$$

dalle që

$$\underline{\underline{\Delta n_k}} = \sqrt{\langle n_k^2 \rangle - \langle n_k \rangle^2} = \underline{\underline{\sqrt{\langle n_k \rangle (1+\langle n_k \rangle)}}}$$

d)

i) Fermionne

$$\frac{\Delta n_k}{\langle n_k \rangle} = \sqrt{\frac{1-\langle n_k \rangle}{\langle n_k \rangle}}$$

Maksimalverdien til $\langle n_k \rangle$ for fermioner er $\langle n_k \rangle = 1$. Altså

$$\frac{\Delta n_k}{\langle n_k \rangle} \to 0 \quad \text{når} \quad \langle n_k \rangle \to 1.$$

ii) Bosoner

$$\frac{\Delta n_k}{\langle n_k \rangle} = \sqrt{\frac{1 + \langle n_k \rangle}{\langle n_k \rangle}}$$

Maksimalverdien til $\langle n_k \rangle = \infty$. Altså

$$\frac{\Delta n_k}{\langle n_k \rangle} \to 1 \quad \text{når} \quad n_k \to \infty.$$

e)

Den søkte sannsynligheten er

$$P(n_1, n_2) = P(0, N) = p(n_1 = 0) \cdot p(n_2 = N)$$

Fra pkt. c) har vi:

$$p(n_k) = \left(1 - e^{\beta(\mu - \varepsilon_k)}\right) \cdot e^{\beta(\mu - \varepsilon_k) n_k}$$

Dette gir

$$p(M_1=0) = (1-e^{\beta\mu})$$

$$p(M_2=N) = (1-e^{\beta(\mu-\varepsilon)})\cdot e^{\beta(\mu-\varepsilon)N}$$

Altså:

$$\underline{\underline{P(M_1=0;\,M_2=N) = (1-e^{\beta\mu})(1-e^{\beta(\mu-\varepsilon)})\,e^{\beta(\mu-\varepsilon)N}}}$$

f) Frå pkt a) har vi:

$$\langle M_1\rangle = \frac{1}{e^{-\beta\mu}-1}$$

$$\langle M_2\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$

Vidare gjeld

$$\langle M_1\rangle + \langle M_2\rangle = \langle N\rangle$$

og

$$\frac{\langle M_1\rangle}{\langle M_2\rangle} = 2$$

Dei to siste likningane gir

$$\langle M_1\rangle = \frac{2}{3}\langle N\rangle \quad \text{og} \quad \langle M_2\rangle = \frac{1}{3}\langle N\rangle$$

Sidan $\langle N\rangle \gg 1$, må $\beta\mu$ og $\beta(\mu-\varepsilon)$ vera nær

Hvorleikar: Då gjeld

$$\langle n_1 \rangle \cong \frac{-1}{\beta\mu} = \frac{2}{3}\langle N \rangle$$

$$\Rightarrow \mu = -\frac{3}{2\beta\langle N \rangle}$$

og

$$\langle n_2 \rangle \cong \frac{1}{\beta(\varepsilon - \mu)} = \frac{1}{3}\langle N \rangle$$

Set inn for μ i denne likninga og får:

$$\frac{1}{\beta\left(\varepsilon + \frac{3}{2\beta\langle N \rangle}\right)} = \frac{1}{3}\langle N \rangle$$

eller

$$\frac{1}{3}\beta\varepsilon\langle N \rangle + \frac{1}{2} = 1$$

$$\Rightarrow \quad \beta = \frac{1}{kT} = \frac{3}{2\varepsilon\langle N \rangle}$$

$$\Rightarrow \quad \underline{\underline{T}} = \underline{\underline{\frac{2\varepsilon\langle N \rangle}{3k}}}$$