TFY Statistisk fysikk
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LF

1a

$$M \equiv \sum_i \langle \sigma_i \rangle$$

$$H = -J \sum_i \sigma_i \sigma_{i+1} - h \sum_i \sigma_i$$

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H}$$

$$= \sum_{\{\sigma_i\}} e^{\beta \left(J \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i\right)}$$

$$M = \frac{1}{Z} \sum_{\{\sigma_i\}} \left(\sum_i \sigma_i\right) e^{\beta \sum_i \left(J \sigma_i \sigma_{i+1} + h \sigma_i\right)}$$

$$= \frac{1}{Z} \sum_{\{\sigma_i\}} \frac{\partial}{\partial(\beta h)} e^{\beta \sum_i \left(J \sigma_i \sigma_{i+1} + h \sigma_i\right)}$$

$$= \frac{1}{Z} \frac{\partial}{\partial(\beta h)} Z = \underline{\left(\frac{\partial \ln Z}{\partial(\beta h)}\right)_T}$$

b)
$$Z = \lambda_+^N + \lambda_-^N$$

$$\lambda_\pm = e^K \left(\cosh w \pm \sqrt{\quad} \right)$$

$$M = \left(\frac{\partial \ln Z}{\partial (\beta h)} \right)_T$$

$$= \frac{\partial}{\partial (\beta h)} \left(\ln \left(\lambda_+^N + \lambda_-^N \right) \right)$$

$$= \frac{N}{\lambda_+^N + \lambda_-^N} \left(\lambda_+^{N-1} \frac{\partial \lambda_+}{\partial \beta h} + \lambda_-^{N-1} \frac{\partial \lambda_-}{\partial \beta h} \right)$$

$$= \frac{N}{\lambda_+^N + \lambda_-^N} \left\{ \lambda_+^{N-1} e^K \left(\sinh w + \frac{\sinh w \cosh w}{\sqrt{\quad}} \right) \right.$$

$$\left. + N \lambda_-^{N-1} e^K \left(\sinh w - \frac{\sinh w \cosh w}{\sqrt{\quad}} \right) \right\}$$

$$= \frac{N \sinh w}{\lambda_+^N + \lambda_-^N} \frac{1}{\sqrt{\quad}} \left(\lambda_+^N - \lambda_-^N \right)$$

$$= \frac{N \sinh w}{\sqrt{\quad}} \left(\frac{\lambda_+^N - \lambda_-^N}{\lambda_+^N + \lambda_-^N} \right)$$

$N \to \infty$:

$$m \equiv \frac{M}{N} = \frac{\sinh w}{\sqrt{\quad}}$$

c)

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$$\tau_i \equiv \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_{i+2} = \tau_i \tau_{i+1}$$

$$H = -\sum_i \left(J_1 \tau_i + J_2 \tau_i \tau_{i+1} \right)$$

$J_1 \leftrightarrow h$, $J_2 \leftrightarrow J$ from 1a, 1b

$\langle \sigma_i \sigma_{i+1} \rangle = \langle \tau_i \rangle \leftrightarrow \langle \sigma_i \rangle$ from 1a

$$\langle \sigma_i \sigma_{i+1} \rangle = \frac{\sinh \omega}{\sqrt{\sinh^2 \omega + e^{-4K}}}$$

$$\omega \equiv \beta J_1$$

$$K \equiv \beta J_2$$

$$\langle \sigma_i \sigma_{i+2} \rangle = \langle \tau_i \tau_{i+1} \rangle$$

$$= \frac{1}{N} \sum_i \langle \tau_i \tau_{i+1} \rangle$$

$$= \frac{1}{N} \frac{1}{Z} \frac{\partial}{\partial(\beta J_2)} Z$$

$$= \frac{1}{N} \frac{\partial \ln Z}{\partial(\beta J_2)}$$

④

Use result for $Z \simeq \lambda_+^N$

$$\lambda_+ = e^K \left(\cosh\omega + \sqrt{}\right)$$

$$\omega = \beta J_1$$

$$K = \beta J_2$$

$$\langle \sigma_i \sigma_{i+2} \rangle = \frac{1}{N} \frac{\partial \ln Z}{\partial K}$$

$$= \frac{\partial \ln \lambda_+}{\partial K}$$

$$= \frac{1}{\lambda_+} \frac{\partial \lambda_+}{\partial K}$$

$$= \frac{1}{\lambda_+} \left(\lambda_+ - \frac{2 e^{-3K}}{\sqrt{}} \right)$$

$$= \underline{\underline{1 - \frac{2 e^{-3K}}{\lambda_+ \sqrt{}}}}$$

i) $T \to 0$; $\beta \to \infty$

$\omega \to \infty$

$K \to \infty$

$\langle \sigma_i \sigma_{i+1} \rangle \longrightarrow 1$

$\langle \sigma_i \sigma_{i+2} \rangle \longrightarrow 1$

Spins are aligned at $T \to 0$.

ii) $T \to \infty$; $\beta \to 0$

$\omega \to 0$

$K \to 0$

$\lambda_+ \to 2$

$\langle \sigma_i \sigma_{i+1} \rangle \longrightarrow 0$

$\langle \sigma_i \sigma_{i+2} \rangle \longrightarrow 0$

At high T, spins are oriented randomly with respect to each other $\Longrightarrow$ no correlations.

2

$$Z = \frac{1}{N!\, h^{dN}} \int d\vec{p}_1 \dots d\vec{p}_N \int d\vec{r}_1 \dots d\vec{r}_N \; e^{-\beta H}$$

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \alpha |\vec{r}_i|^d$$

b) Momentum integrals and spatial integrals factorize.

$$Z = \frac{1}{N!\, h^{dN}} (Z_1)^N$$

$$Z_1 = \underbrace{\int d\vec{p}_1 \, e^{-\beta \frac{\vec{p}_1^{\,2}}{2m}}}_{= (2\pi m k_B T)^{\frac{d}{2}}} \cdot \int d\vec{r}_1 \, e^{-\beta\alpha |\vec{r}_1|^d}$$

$$\int d\vec{r}_1 \; e^{-\beta\alpha |\vec{r}_1|^d}$$

$$= \Omega_d \int_0^R dr \; r^{d-1} \; e^{-\beta\alpha r^d}$$

$$= \frac{\Omega_d}{d} \int_0^{R^d} du \; e^{-\beta\alpha u}$$

$$= \frac{\Omega_d}{\beta\alpha d} \int_0^{\beta\alpha R^d} dx \; e^{-x}$$

$$= \frac{\Omega_d}{\beta\alpha d} \left(1 - e^{-\beta\alpha R^d}\right)$$

$$= \underbrace{\frac{\Omega_d R^d}{d}}_{\equiv V} \; \frac{1}{\underbrace{\beta\alpha R^d}_{= x}} \left(1 - e^{-\beta\alpha R^d}\right)$$

$$= V \left(\frac{1 - e^{-x}}{x}\right)$$

$$Z = \frac{1}{N! \, h^{dN}} \; \frac{V^N}{\left(\frac{1}{\sqrt{2\pi m k_B T}}\right)^{dN}} \left(\frac{1 - e^{-x}}{x}\right)^N$$

$$= \frac{V^N}{N! \, \Lambda^{dN}} \left(\frac{1 - e^{-x}}{x}\right)^N$$

b)

$$U = -\frac{\partial}{\partial\beta} \ln Z$$

$$= \frac{dN}{2}\cdot k_B T - N\frac{\partial}{\partial\beta}\ln\left(\frac{1-e^{-x}}{x}\right)$$

$$= \frac{dN}{2}\cdot k_B T - N\underbrace{\frac{\partial}{\partial x}\ln\left(\frac{1-e^{-x}}{x}\right)}_{=\left(\frac{1}{e^{x}-1}-\frac{1}{x}\right)}\underbrace{\frac{\partial x}{\partial\beta}}_{=\frac{d\alpha V}{\Omega d}}$$

$$= \frac{dN}{2}\cdot k_B T + N\frac{d\alpha V}{\Omega d}\left(\frac{1}{x}-\frac{1}{e^{x}-1}\right)$$

$$S = \frac{1}{T}(U-F)$$

U given above

$$F = -k_B T \ln Z$$

$$= -k_B T\left(-\ln(N!) + N\ln V - Nd\ln\Lambda + N\ln\left(\frac{1-e^{-x}}{x}\right)\right)$$

3)

$$\langle N\rangle = \frac{\partial}{\partial(\beta\mu)} \ln Z_g$$

$$= \sum_{\vec{k},\sigma} \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}$$

$$U = \sum_{k,\sigma} \frac{\varepsilon_k}{e^{\beta(\varepsilon_k-\mu)}+1}$$

$$\langle N\rangle = 2\cdot \int_0^\infty d\varepsilon\, g(\varepsilon) \frac{1}{e^{\beta\varepsilon}\frac{1}{z}+1}$$

$$= 2\int_0^\infty d\varepsilon\, g(\varepsilon)\, z\, \frac{e^{-\beta\varepsilon}}{1+z e^{-\beta\varepsilon}}$$

$$= 2\int_0^\infty d\varepsilon\, g(\varepsilon)\, z \cdot e^{-\beta\varepsilon} \sum_{l=0}^\infty (-1)^l z^l e^{-\beta\varepsilon l}$$

$$= 2\sum_{l=0}^\infty (-1)^l z^{l+1} \int_0^\infty d\varepsilon\, e^{-\beta\varepsilon(l+1)} \cdot \frac{V 2\pi\varepsilon}{(2\pi\hbar c)^2}$$

(10)

$$= \frac{4\pi V}{(2\pi\hbar c)^2} \sum_{l=0}^{\infty} (-1)^l z^{l+1} \frac{1}{(\beta(l+1))^2} \underbrace{\int_0^{\infty} dx\, x e^{-x}}_{=1}$$

$$= \frac{4\pi V}{(2\pi\hbar c)^2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{\beta^2 l^2} z^l$$

$$= V \sum_{l=1}^{\infty} l\, b_l\, z^l \; ; \; b_l = \frac{4\pi}{(2\pi\hbar c)^2} \frac{(-1)^{l+1}}{\beta^2 l^3}$$

$$U = 2 \int_0^{\infty} d\varepsilon\, g(\varepsilon)\, z \frac{\varepsilon\, e^{-\beta\varepsilon}}{1 + z e^{-\beta\varepsilon}}$$

$$= \frac{4\pi V}{(2\pi\hbar c)^2} \cdot \sum_{l=0}^{\infty} (-1)^l z^{l+1} \underbrace{\int_0^{\infty} d\varepsilon\, \varepsilon^2 e^{-\beta\varepsilon(l+1)}}_{= \frac{1}{\beta^3 (l+1)^3} \underbrace{\int_0^{\infty} dx\, x^2 e^{-x}}_{=2}}$$

$$\frac{\beta U}{2} = \frac{4\pi V}{(2\pi\hbar c)^2} \sum_{l=1}^{\infty} (-1)^{l+1} \frac{z^l}{\beta^2 l^3}$$

$$\frac{\beta U}{2} = V \sum_{l=1}^{\infty} b_l z^l$$

$$b_l = \frac{4\pi}{(2\pi\hbar c)^3} \frac{(-1)^{l+1}}{\beta^2 l^3}$$

$$\beta p V = \ln Z_g$$

$$\langle N \rangle = z \frac{\partial \ln Z}{\partial z} = V \sum_{l=1}^{\infty} l b_l z^l$$

$$\Rightarrow \beta p V = V \sum_{l=1}^{\infty} b_l z^l = \frac{\beta U}{2}$$

$$\frac{U}{pV} = 2$$

5) Pressure at $T = 0$

$$pV = \frac{U}{2}$$

$$= \frac{1}{2} \sum_{\vec{k},\sigma} \frac{\varepsilon_k}{e^{\beta(\varepsilon_k - \mu)} + 1}$$

$$= \frac{1}{2} \cdot 2 \cdot \int_0^{\mu} d\varepsilon \, g(\varepsilon) \, \varepsilon$$

$$pV = \frac{V}{(2\pi\hbar c)^2} \cdot 2\pi \underbrace{\int_0^\mu d\varepsilon \, \varepsilon^2}_{\frac{\mu^3}{3}}$$

$$\langle N \rangle = \frac{V}{(2\pi\hbar c)^2} \cdot 2\pi \cdot \underbrace{\int_0^\mu d\varepsilon \, \varepsilon}_{\frac{\mu^2}{2}}$$

$$\frac{\langle N \rangle}{V} \frac{(2\pi\hbar c)^2}{2\pi} \, 2 = \mu^2$$

$$\mu = \frac{1}{\sqrt{\pi}} \, 2\pi\hbar c \, \rho$$

$$pV = \frac{V}{(2\pi\hbar c)^2} \frac{2\pi}{\pi^{3/2}} (2\pi\hbar c)^3 \rho^3$$

$$p = \frac{2\pi}{\pi^{3/2}} \, 2\pi\hbar c \, \rho^3$$

$$\hbar \to 0 \Rightarrow p = 0$$

c)

$$\langle N_{\sigma=1} \rangle = \frac{\langle N \rangle}{2} = \langle N_{\sigma=-1} \rangle$$

$$m = \frac{1}{N}\left(\langle N_{\sigma=1} \rangle - \langle N_{\sigma=-1} \rangle \right) = 0$$