Consider a random walk on the x-axis, the probability of going to the left and right are equal. The step-length increases with each step such that the position after N steps is:

$$X = \sum_{i=1}^{N} i\Delta_i = \Delta_1 + 2\Delta_2 + 3\Delta_3 + \cdots + N\Delta_N.$$

The random variables Δ_i can take the values $\Delta_i = \pm 1$ and they are uncorrelated: $\langle \Delta_i \Delta_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta. The mean square displacement after N steps is:

Select one alternative:

- $\langle X^2 \rangle = N$
- $\langle X^2 \rangle = \sum_{i=1}^{N} i^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2$
- $\langle X^2 \rangle = N^4$
- $\langle X^2 \rangle = N^2$
- $\langle X^2 \rangle = \sum_{i=1}^{N} i = 1 + 2 + 3 + \cdots + N$
- $\langle X^2 \rangle = \sqrt{N}$

Solution 1

$$\langle X^2 \rangle = \left\langle \sum_{i=1}^{N} \sum_{j=1}^{N} i\Delta_i \, j\Delta_j \right\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} ij \langle \Delta_i \Delta_j \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} ij\delta_{ij} = \sum_{i=1}^{N} i^2$$

Consider a particle in one dimension.
The total energy is: $E = \frac{p_x^2}{2m} + Kx$, where $K > 0$ is a constant, x the position, and p_x the momentum.
The allowed positions of the particle are on the positive x-axis: $0 \leq x \leq \infty$.
We are in the microcanonical ensemble where the total energy is constant, and the probability distribution is:
$P(p_x, x) = C\,\delta\left[\frac{p_x^2}{2m} + Kx - E\right]$
where C is a constant.
Energy conservation restricts the momentum to the interval: $-\sqrt{2mE} \leq p_x \leq \sqrt{2mE}$.
The probability distribution for p_x on this interval is:

Select one alternative:

- $P(x) = C\left(1 - \frac{p_x^2}{2mE}\right)$
- $P(p_x) = \frac{C}{K} e^{-\frac{p_x^2}{2mE}}$
- $P(p_x) = C\delta\left[\frac{p_x^2}{2m} - E\right]$
- $P(p_x) = \frac{C}{K}$
- $P(p_x) = \frac{C}{K}\frac{p_x^2}{2mE}$
- $P(p_x) = \frac{C}{p_x}$

Solution 2

$$
\begin{aligned}
P(p_x) &= \int_0^\infty P(p_x, x)\, \mathrm{d}x \\
&= \int_0^\infty C\delta\left[Kx - (E - \frac{p_x^2}{2m})\right]\, \mathrm{d}x \\
&= \int_0^\infty C\,\delta\left[\tilde{x} - (E - \frac{p_x^2}{2m})\right]\, \frac{1}{K}\mathrm{d}\tilde{x} \\
&= \frac{C}{K}
\end{aligned}
$$

A system has energy levels $E = n\epsilon$, where $n = 0, 1, 2, 3, \cdots, \infty$ and $\epsilon > 0$ a constant. At the energy level n there are $\Gamma_n = n^2$ number of states of the system. We are in the microcanonical ensemble and $\frac{dS}{dE} = \frac{1}{T}$. Consider that n is large and ϵ small. Express the entropy as a function of energy $S = S(E)$. What is the relation between energy and temperature in the system?

.

Select one alternative:

- $E = k_B T$
- $E = \frac{\epsilon^2}{k_B T}$
- $E = 2k_B T$
- $E = k_B T\, \mathrm{e}^{-\frac{\epsilon}{k_B T}}$
- $E = \frac{(k_B T)^2}{\epsilon}$
- $E = \epsilon\, \mathrm{e}^{-\frac{\epsilon}{k_B T}}$

Solution 3

$$S = k_B \ln \Gamma_n = k_B \ln\left(n^2\right) = k_B \ln\left(\frac{E^2}{\epsilon^2}\right)$$

$$\frac{dS}{dE} = \frac{1}{T} \;\Rightarrow\; \frac{2k_B}{E} = \frac{1}{T} \;\Rightarrow\; E = 2k_B T$$

Consider a particle in two dimensions confined inside a recipient with area A.
The particle has the Hamiltonian: $H = \frac{p_x^2+p_y^2}{2m} + U(x,y)$. We are in the canonical ensemble. Let the potential energy be a constant: $U(x,y) = a$.
The partition function is:

Select one alternative:

- $Z = \frac{2\pi m}{h^2} k_B T A \mathrm{e}^{-\frac{a}{k_B T}}$
- $Z = \frac{h^2}{2\pi m} (k_B T)^2 A \mathrm{e}^{-2\frac{a}{k_B T}}$
- $Z = (\frac{2\pi m}{h^2})^{3/2} k_B T A \mathrm{e}^{-\frac{a}{k_B T}}$
- $Z = \frac{2\pi m}{h^2} (k_B T)^2 A \mathrm{e}^{-\frac{a}{k_B T}}$
- $Z = \frac{2\pi m}{h^2} k_B T A^2 \mathrm{e}^{\frac{a}{k_B T}}$
- $Z = \frac{2\pi m}{h^2} \frac{A}{k_B T} \mathrm{e}^{\frac{a}{k_B T}}$

Solution 4

$$
\begin{aligned}
Z &= \frac{1}{h^2} \int \mathrm{d}p_x \int \mathrm{d}p_y \int \mathrm{d}x \int \mathrm{d}y \, \mathrm{e}^{-\beta \frac{p_x^2+p_y^2}{2m} - \beta a} \\
&= \frac{1}{h^2} \frac{\pi 2m}{\beta} A \mathrm{e}^{-\beta a} \\
&= \frac{2\pi m}{h^2} k_B T A \mathrm{e}^{-\beta a}
\end{aligned}
$$

A system has the density of states: $\rho(E) = C\mathrm{e}^{-E/E_0}$ with $E \geq 0$ and E_0 and C are positive constants. We are in the canonical ensemble and the partition function is $Z = \int_0^\infty dE\,\rho(E)e^{-\beta E}$.
The mean energy is:

Select one alternative:

- $\langle E \rangle = E_0$
- $\langle E \rangle = 0$
- $\langle E \rangle = \frac{(k_B T)^2}{E_0}$
- $\langle E \rangle = \frac{E_0\, k_B T}{E_0 + k_B T}$
- $\langle E \rangle = k_B T$
- $\langle E \rangle = \frac{E_0^2}{k_B T}$

Solution 5

Partition function:

$$Z = \int_0^\infty C\mathrm{e}^{-E/E_0}\mathrm{e}^{-\beta E}\mathrm{d}E = \frac{C}{\frac{1}{E_0} + \beta}$$

Mean energy:

$$\langle E \rangle = -\frac{\partial}{\partial \beta}\ln Z = \frac{\partial}{\partial \beta}\ln\left(\frac{1}{E_0} + \beta\right) = \frac{1}{\frac{1}{E_0} + \beta} = \frac{E_0 k_B T}{E_0 + k_B T}$$

Consider a single particle inside a container of volume V. The container is divided into two parts of equal volume $\frac{V}{2}$, in one side the potential energy of the particle is $U=\epsilon$, in the other part the potential energy is $U=0$. Calculate the average potential energy of the particle $\langle U\rangle$ in the canonical ensemble. At any given instant what is the probability P of finding the particle in the part of the container with energy $U=\epsilon$?

Select one alternative:

- $P=\frac{1}{2}-\frac{\epsilon}{k_B T}$
- $P=\frac{1}{(e^{\beta\epsilon}+1)^2}$
- $P=\frac{1}{e^{\beta\epsilon}-1}$
- $P=\frac{1}{2}-\left(\frac{\epsilon}{k_B T}\right)^2$
- $P=\frac{1}{e^{\beta\epsilon}+1}$
- $P=\frac{1}{2}$

Solution 6

Partition function:

$$
\begin{aligned}
Z &= \frac{1}{h^3}\int \mathrm{d}^3p \int \mathrm{d}^3 r \mathrm{e}^{-\beta\frac{p^2}{2m}-\beta U} \\
&= \left(\frac{2\pi m}{\beta h^2}\right)^{3/2}\left[\frac{V}{2}+\frac{V}{2}\mathrm{e}^{-\beta\epsilon}\right]
\end{aligned}
$$

Mean energy:

$$
\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = \frac{3}{2\beta}+\frac{\epsilon}{\mathrm{e}^{\beta\epsilon}+1} = \frac{3}{2}k_B T+\frac{\epsilon}{\mathrm{e}^{\beta\epsilon}+1}
$$

The mean kinetic energy is $\langle T\rangle = \frac{3}{2}k_B T$, hence :

$$
\langle U\rangle = \frac{\epsilon}{\mathrm{e}^{\beta\epsilon}+1} \Rightarrow P=\frac{1}{\mathrm{e}^{\beta\epsilon}+1}
$$

Consider a particle in one dimension with Hamiltonian $H = \frac{p_x^2}{2m} + \frac{k_B T}{L} x$ and the constraint $0 \leq x \leq \infty$. We are in the canonical ensemble. What is the probability p that the the position of the particle is $x \geq L$?

Select one alternative:

- $p = \frac{1}{4}$
- $p = \frac{1}{2}$
- $p = \mathrm{e}^{-1}$
- $p = 1 - \mathrm{e}^{-2}$
- $p = 1 - \mathrm{e}^{-3}$
- $p = 1 - \mathrm{e}^{-1}$

Solution 7

Canonical distribution:

$$P(p_x, x) = C\,\mathrm{e}^{-\beta H} = C\mathrm{e}^{-\beta \frac{p_x^2}{2m} - \beta k_B T \frac{x}{L}}$$

Which implies:

$$P(x) = \tilde{C}\mathrm{e}^{-\frac{x}{L}}$$

Normalization:

$$\int_0^\infty \mathrm{d}x P(x) = 1 \;\Rightarrow\; P(x) = \frac{1}{L}\mathrm{e}^{-\frac{x}{L}}$$

Probability that $x > L$

$$\int_L^\infty P(x)\mathrm{d}x = \int_L^\infty \frac{1}{L}\mathrm{e}^{-\frac{x}{L}}\mathrm{d}x = \mathrm{e}^{-1}$$

A particle with Hamiltonian $H = \frac{p_x^2}{2m}$ is moving freely (no external forces) in one dimension. The phase space is (x, p_x). Consider a rectangle in phase space at time $t = 0$, defined by the four corners (x, p_x) , $(x + \Delta x, p_x)$, $(x, p_x + \Delta p_x)$, $(x + \Delta x, p_x + \Delta p_x)$. Consider the area defined by all the points inside this rectangle. What happens to this area as time evolves?

Select one alternative:

- The area in phase space is just locally preserved, therefore the area changes since the rectangle has finite size.
- The area in phase space is not preserved. The circumference is preserved.
- The rectangle is simply translated and does not change shape.
- The area is conserved, and the circumference of the area goes toward a constant.
- The area is conserved and the circumference goes to zero.
- The area is conserved, but the rectangle is deformed such that the circumference of the area diverges.

Solution 8

The rectangle is deformed into a parallelogram with the same height and base as the rectangle. This implies area is conserved. At the same time the parallelogram is stretched so that the circumference goes to infinity.

A classical non-relativistic particle in two dimensions is confined inside a circle of radius R. The particle is subject to a potential $U(r) = -\epsilon$ when $r < R_0$, and $U(r) = 0$ when $R_0 < r < R$.
r is the distance from the origin (center of circle). The mass of the particle is m.
The canonical partition function of the particle is:

Select one alternative:

- $Z = \frac{2\pi m}{\beta h^2} \pi R^2 \, \mathrm{e}^{\beta\epsilon}$
- $Z = \left(\frac{2\pi m}{\beta h^2}\right)^{3/2} \pi(R^2 - R_0^2) \, \mathrm{e}^{\beta\epsilon}$
- $Z = \frac{2\pi m}{\beta h^2} \pi(R^2 - R_0^2) \, \mathrm{e}^{\beta\epsilon}$
- $Z = \frac{2\pi m}{\beta h^2} \left[\pi R^2 + \pi R_0^2 \, \mathrm{e}^{\beta\epsilon}\right]$
- $Z = \frac{2\pi m}{\beta h^2} \left[\pi(R^2 - R_0^2) \, \mathrm{e}^{-\beta\epsilon} + \pi R_0^2 \, \mathrm{e}^{\beta\epsilon}\right]$
- $Z = \frac{2\pi m}{\beta h^2} \left[\pi(R^2 - R_0^2) + \pi R_0^2 \, \mathrm{e}^{\beta\epsilon}\right]$

Solution 9

$$
\begin{aligned}
Z &= \frac{1}{h^2} \int \mathrm{d}^2 p \int \mathrm{d}^2 r \mathrm{e}^{-\beta \frac{p_x^2 + p_y^2}{2m} - \beta U} \\
&= \frac{2\pi m}{h^2 \beta} \int \mathrm{d}^2 r \mathrm{e}^{-\beta U} \\
&= \frac{2\pi m}{h^2 \beta} \left[\pi(R^2 - R_0^2) + \pi R_0^2 \mathrm{e}^{\beta\epsilon}\right]
\end{aligned}
$$

An ideal classical two-dimensional gas of N particles is confined inside an area A. All particles have the same mass m. The canonical partition function is:

Select one alternative:

- $Z_N = \frac{1}{N!h^{2N}}(2\pi m k_B T)^N A^N$
- $Z_N = \frac{1}{N!h^{2N}}(2\pi m k_B T)^{-2N} A^{2N}$
- $Z_N = \frac{1}{N!h^{2N}}(\frac{2\pi m}{k_B T})^N A^N$
- $Z_N = \frac{1}{N!h^{2N}}(2\pi m k_B T)^{N/2} A^N$
- $Z_N = \frac{1}{N!h^{2N}}(\frac{2\pi m}{k_B T})^{2N} A^{2N}$
- $Z_N = \frac{1}{N!h^{2N}}(2\pi m k_B T)^N A^{N/2}$

Solution 10

Kinetic energy:

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m}$$

Mean energy:

$$\begin{aligned} Z &= \frac{1}{h^{2N}N!}\int \mathrm{d}^2p_1 \cdots \int \mathrm{d}^2p_N \int \mathrm{d}^2r_1 \cdots \mathrm{d}^2r_N \mathrm{e}^{-\beta H} \\ &= \frac{1}{h^{2N}N!}\left(\int \mathrm{d}^2p_1 \mathrm{e}^{-\beta\frac{\vec{p}_1^2}{2m}}\right)^N \left(\int \mathrm{d}^2r_1\right)^N \\ &= \frac{1}{h^{2N}N!}\left(2\pi m k_B T\right)^N A^N \end{aligned}$$

A system of N classical distinguishable particles can occupy two energy levels, one ground state and one excited state. How many different configurations are there with 2 particles in the ground state?

Select one alternative:

- $\frac{N!}{2!}$
- $N^2 - N$
- $\frac{N(N-1)}{2}$
- $\frac{N}{2}$
- N
- $\frac{N^2}{2}$

Solution 11

The number of ways of placing N particles one two energy levels such that 2 of them are in in the ground state is $\binom{N}{2} = \frac{N!}{(N-2)!2!} = \frac{N(N-1)}{2}$.

Another way of seeing this is to choose one of the particles first, then we have N particles to choose between. When we choose the second particle we have $N-1$ to choose. Which gives $N(N-1)$ possibilities. It does not matter which order we choose them, and we therefore divid by 2, i.e. $N(N-1)/2$.

In classical statistical mechanics we divide by a factor $N!$ when we count the states of N particles (in a gas for example). Which one of these statements is correct?

Select one alternative:

- The factor N! is only used in the micro-canonical ensemble.
- The factor N! is only important at low temperatures.
- The factor N! is not important when calculating the grand partition function of a system.
- The factor N! makes entropy finite both in the canonical and micro-canonical ensemble.
- The factor N! makes entropy an extensive quantity both in the canonical and micro-canonical ensemble.
- The factor N! is only used in the canonical ensemble.

Solution 12

The factor $N!$ makes entropy an extensive property both in the canonical and micro-canonical ensemble.

Which one of these statements is correct for the Debye model for the heat capacity of solids?

Select one alternative:

- The heat capacity in the low temperature limit comes mostly from the long wavelength elastic deformations (phonons) in the solid.
- The heat capacity in the low temperature limit is the sum of the heat capacity of individual atoms, interactions between the atoms can be neglected.
- The model is only valid in the low temperature limit.
- The heat capacity in the high temperature limit comes mostly from the long wavelength elastic deformations (phonons) in the solid.
- The model is only valid in the high temperature limit.
- The density of states is a delta-function in frequency.

Solution 13

The heat capacity in the low temperature limit comes mostly from long wavelength elastic deformations (phonons) in the solid.

A system with N particles has the canonical partition function $Z = \mathrm{e}^{-\beta\epsilon N}$.
The variance of the energy $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$ is:

Select one alternative:

- $\Delta E^2 = \epsilon^2$
- $\Delta E^2 = \frac{\epsilon^2}{N}$
- $\Delta E^2 = \sqrt{N}\epsilon^2$
- $\Delta E^2 = N\epsilon^2$
- $\Delta E^2 = N^2\epsilon^2$
- $\Delta E^2 = 0$

Solution 14

Easiest solution: This is the partition function of a system with only one energy level. There can therefore be no fluctuations in energy, i.e. $\Delta E^2 = 0$.

Direct calculation:

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z = \frac{\partial}{\partial\beta}\beta\epsilon N = \epsilon N$$

$$\Delta E^2 = -\frac{\partial}{\partial\beta}\langle E \rangle = -\frac{\partial}{\partial\beta}\epsilon N = 0$$

Consider a system of three spins with Hamiltonian: $H = -Js_1s_2 - 2Js_2s_3$, where J is a postive constant, and $s_i = \pm 1$. We are in the canonical ensemble. In the high temperature limit $k_BT \gg J$ the free energy $F = \langle E \rangle - TS$ is:

Select one alternative:

- $F \approx -3J$
- $F \approx -\ln(2)k_BT$
- $F \approx -3\ln(2)k_BT$
- $F \approx \frac{3}{2}k_BT$
- $F \approx -\frac{(k_BT)^2}{J}$
- $F \approx 3k_BT$

Solution 15

Method 1: when T is large the average energy $\langle E \rangle$ goest to zero, since the spins have random orientations. Then number of states is 2^3 when the spins are random, i.e the entropy is $S = k_B \ln 2^3 = 3k_B \ln 2$. Which gives the free energy $F \approx \langle E \rangle - TS = -3\ln(2)k_BT$ at high temperature.

Method 2:

$$
\begin{aligned}
Z &= \sum_{s_1,s_2,s_3} \mathrm{e}^{\beta J(s_1s_2+2s_2s_3)} \\
&= \sum_{s_1,s_2} \mathrm{e}^{\beta J s_1 s_2} \left(\mathrm{e}^{2\beta J s_2} + \mathrm{e}^{-2\beta J s_2}\right) \\
&= \sum_{s_1,s_2} \mathrm{e}^{\beta J s_1 s_2} \left(\mathrm{e}^{2\beta J} + \mathrm{e}^{-2\beta J}\right) \\
&= 2\left(\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J}\right)\left(\mathrm{e}^{2\beta J} + \mathrm{e}^{-2\beta J}\right)
\end{aligned}
$$

When $T \gg k_BT$ we have $\beta J \ll 1$, and $Z \approx 2^3$ Free energy:

$$F = -k_BT \ln Z \approx -k_BT \ln(2^3) = -k_BT 3 \ln(2)$$

Consider a system of three spins with Hamiltonian: $H = -Js_1s_2 + Js_2s_3$, where J is a postive constant, and $s_i = \pm 1$. We are in the canonical ensemble. When $T \to 0$ the mean energy $\langle E \rangle$ and heat capacity C become:

Select one alternative:

- $\langle E \rangle = 0$ and $C = k_B$
- $\langle E \rangle = -2J$ and $C = k_B$
- $\langle E \rangle = -2J$ and $C = 0$
- $\langle E \rangle = 0$ and $C = 0$
- $\langle E \rangle = -2J$ and $C = 2k_B$
- $\langle E \rangle = -J$ and $C = 2k_B$

Solution 16

The minimum energy is obtained when the spins s_1 and s_2 are parallel, and s_2 and s_3 are anti-parallel. I.e. the mininum energy is $E = -2J$, which is the mean energy at zero temperature. When $T = 0$ the spins are locked in the same position, there is no thermal fluctuations that can flip the spins (degree of freedom frozen out). I.e. we have $C = 0$.

A system is described by the Hamiltonian $H = J(s_1 s_2 + s_2 s_3 + s_3 s_1)$ where $J > 0$ and the spins take the values $s_i = \pm 1$. When $T \to 0$ the mean energy and entropy become:

Select one alternative:

- $\langle E \rangle = -2J$ and $S > 0$
- $\langle E \rangle = -3J$ and $S > 0$
- $\langle E \rangle = -2J$ and $S = 0$
- $\langle E \rangle = -J$ and $S = 0$
- $\langle E \rangle = -J$ and $S > 0$
- $\langle E \rangle = -3J$ and $S = 0$

Solution 17

There are $2^3 = 8$ spin configurations. If all the spins point in the same direction, $(s_1, s_2, s_3) = (1, 1, 1)$ or $(s_1, s_2, s_3) = (-1, -1, -1)$, the energy is $H = 3J$. The only other configurations available is when two spins are parallel and one anti-parallel with the two others, for example $(s_1, s_2, s_3) = (1, 1, -1)$ or $(s_1, s_2, s_3) = (-1, 1, -1)$, in which case the energy is $H = -J$, the minimum energy in the system. Since there is more than one state with this energy, the entropy is non-zero. I.e at $T = 0$ we have $\langle E \rangle = -J$ and $S > 0$.

A system of spins $s_i = \pm 1$ is such that half of the spins align and the other half anti-align when subject to an external field h. The Hamiltonian is
$H = -h(s_1 + s_2 + \cdots + s_N) + h(s_{N+1} + s_{N+2} + \cdots + s_{2N})$
with $h > 0$. We are in the canonical ensemble. The free energy of the system is:

Select one alternative:

- $F = -2N\frac{(k_BT)^2}{h}$
- $F = -\frac{N}{\beta}\ln\left(e^{2\beta h} + e^{-2\beta h}\right)$
- $F = -\frac{2N}{\beta}\ln\left(e^{\beta h} + e^{-\beta h}\right)$
- $F = -2N\frac{h^2}{k_BT}$
- $F = -\frac{2N}{\beta}\ln\left(1 + e^{2\beta h}\right)$
- $F = -\frac{2N}{\beta}\ln\left(1 + e^{-2\beta h}\right)$

Solution 18

Partition function:

$$\begin{aligned}
Z &= \sum_{s_1=\pm1}\cdots\sum_{s_{2N}=\pm1} e^{\beta h(s_1+s_2+\cdots+s_N)-\beta h(s_{N+1}+s_{N+2}+\cdots+s_{2N})} \\
&= \left(\sum_{s_1=\pm1} e^{\beta h s_1}\right)^N \left(\sum_{s_{N+1}=\pm1} e^{-\beta h s_{N+1}}\right)^N \\
&= \left(e^{\beta h} + e^{-\beta h}\right)^N \left(e^{-\beta h} + e^{\beta h}\right)^N
\end{aligned}$$

Free energy:

$$F = -\frac{1}{\beta}\ln Z = -\frac{2N}{\beta}\ln\left(e^{\beta h} + e^{-\beta h}\right)$$

A system has the Hamiltonian $H = -J\phi_1^2\phi_2$, where $J > 0$ is a constant and the variables ϕ_i can take two values $\phi_i = 1$ or $\phi_i = -1$. We are in the canonical ensemble. The average value of $\phi_1^2\phi_2$ is:

Select one alternative:

- $\langle\phi_1^2\phi_2\rangle = \mathrm{e}^{-\beta J}$
- $\langle\phi_1^2\phi_2\rangle = \frac{\mathrm{e}^{\beta J} - \mathrm{e}^{-\beta J}}{\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J}}$
- $\langle\phi_1^2\phi_2\rangle = 1 - \mathrm{e}^{-2\beta J}$
- $\langle\phi_1^2\phi_2\rangle = 1 - \frac{1}{1+(\beta J)^2}$
- $\langle\phi_1^2\phi_2\rangle = \frac{\mathrm{e}^{2\beta J} - \mathrm{e}^{-2\beta J}}{\mathrm{e}^{2\beta J} + \mathrm{e}^{-2\beta J}}$
- $\langle\phi_1^2\phi_2\rangle = 1 - \mathrm{e}^{-\beta J}$

Solution 19

$$Z = \sum_{\phi_1=\pm 1}\sum_{\phi_2=\pm 1} \mathrm{e}^{\beta J \phi_1^2 \phi_2} = 2\left(\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J}\right)$$

Mean energy:

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = -J\,\frac{\mathrm{e}^{\beta J} - \mathrm{e}^{-\beta J}}{\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J}}$$

$$\langle E\rangle = -J\langle\phi_1^2\phi_2\rangle \quad\Rightarrow\quad \langle\phi_1^2\phi_2\rangle = \frac{\mathrm{e}^{\beta J} - \mathrm{e}^{-\beta J}}{\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J}}$$

A paramagnet is given by the Hamiltonian:

$$H = \sum_{i=1}^{N}(-hs_i + \epsilon s_i^2) = -h(s_1 + s_2 + \cdots + s_N) + \epsilon(s_1^2 + s_2^2 + \cdots + s_N^2)$$

where $s_i = \pm 1$, and h and ϵ are positive constants, and $\epsilon \ll h$. We are in the canonical ensemble. When $T = 0$ the entropy is :

Select one alternative:

- $S = k_B \ln 2$
- $S = Nk_B$
- $S = 0$
- $S = Nk_B \frac{\epsilon}{h}$
- $S = Nk_B \ln 2$
- $S = N\frac{k_B}{2}$

Solution 20

When $T = 0$ the system is in a minimum energy state. The minimum of $-hs_1 + \epsilon s_1^2$ is when $s_1 = 1$. The same applies of course to all the other spins. There is only one ground state, hence the entropy is $S = 0$.

N non-interacting vector spins in the x-y plane are oriented by an external field $\vec{h}$. The Hamiltonian is:
$H = -\vec{h} \cdot (\vec{s}_1 + \vec{s}_2 + \cdots + \vec{s_N})$
where $\vec{s}_i = (\cos\theta_i, \sin\theta_i)$ with $-\pi \leq \theta_i \leq \pi$. The field $\vec{h} = (h, 0) = h\vec{e}_x$ is oriented along the x-axis. The mean energy in the low temperature limit $k_B T \ll h$ is:

Select one alternative:

- $\langle E \rangle \approx -Nh + N\frac{(k_B T)^2}{h}$
- $\langle E \rangle \approx -Nh - Nk_B T$
- $\langle E \rangle \approx -Nh + N\frac{(k_B T)^3}{h^2}$
- $\langle E \rangle \approx \frac{N}{2} k_B T$
- $\langle E \rangle \approx -Nh + \frac{N}{2} k_B T$
- $\langle E \rangle \approx N\frac{(k_B T)^3}{h^2}$

Solution 21
The Hamiltonian is:

$$H = -h\cos(\theta_1) - h\cos(\theta_2) - \cdots - h\cos(\theta_N)$$

In the low temperature limit θ_i is close to zero, and we can do a series expansion:

$$H \approx -h\left(1 - \frac{\theta_1^2}{2}\right) - h\left(1 - \frac{\theta_2^2}{2}\right) - \cdots - h\left(1 - \frac{\theta_N^2}{2}\right)$$

The equipartition theorem then gives:

$$\langle H \rangle \approx -Nh + N\frac{1}{2}k_B T$$

Consider a system that has free energy $F = k_BTN - k_BTN\ln(N)$ where N is the number of particles in the system. The volume is constant. The system is connected to a reservoir of N_r particles with free energy $F_r = N_r\mu$, where $\mu < 0$ is a constant chemical potential. Assume that N is macroscopically large and that we always have $N_r \gg N$. What is the number of particles in the system in thermal equilibrium?

Select one alternative:

- $N = \exp\left(-\frac{\mu}{k_BT}\right)$
- $N = -\frac{\mu}{k_BT}$
- $N = (\frac{\mu}{k_BT})^2$
- $N = \exp\left(-\frac{\mu}{k_BT}\right) + \exp\left(-\frac{2\mu}{k_BT}\right)$
- $N = -\frac{\mu}{k_BT}\exp\left(-\frac{\mu}{k_BT}\right)$
- $N = \frac{2}{\exp\left(-\frac{\mu}{k_BT}\right)-1}$

Solution 22

Chemical potential of the system:

$$\mu_s = \frac{\partial F}{\partial N} = -k_BT\ln(N)$$

In thermal equilibrium the chemical potential of the system and the reservoir must be equal:

$$\mu_s = \mu \;\;\Rightarrow\;\; -k_BT\ln(N) = \mu \;\;\Rightarrow\;\; N = \exp\left(-\frac{\mu}{k_BT}\right)$$

Consider a classical two-dimensional ideal gas of N molecules in a container of constant area A. The chemical potential of the molecules in the gas is:
$\mu = k_B T \ln(N\lambda^2/A)$, where λ is the thermal de Broglie length.
The molecules can adsorb onto the walls of the container. The gas molecules adsorbed on the walls have chemical potential:
$\mu_s = -\epsilon + k_B T \ln(N_s\lambda^2/A_s)$, where $\epsilon > 0$, N_s is the number of adsorbed molecules, and A_s the surface area in which the molecules adsorb. The total number of molecules in the container is $N_t = N + N_s$. In equilibrium the mean number of molecules on the surface can be expressed as:

Select one alternative:

- $N_s = N\frac{A_s}{A}$
- $N_s = N\exp\left(\frac{\epsilon}{k_B T}\right)$
- $N_s = N\frac{A_s^2}{A^2}\exp\left(\frac{\epsilon}{k_B T}\right)$
- $N_s = \frac{N}{1+\frac{A_s}{A}\exp\left(-2\frac{\epsilon}{k_B T}\right)}$
- $N_s = \frac{N}{1+\exp\left(-\frac{\epsilon}{k_B T}\right)}$
- $N_s = N\frac{A_s}{A}\exp\left(\frac{\epsilon}{k_B T}\right)$

Solution 23

In thermal equilbrium we must have

$$\mu = \mu_s$$

Which implies:

$$k_B T \ln\left(\frac{N\lambda^2}{A}\right) = -\epsilon + k_B T \ln\left(\frac{N_s\lambda^2}{A_s}\right)$$

Dividing by $k_B T$ and taking the exponential of both sides gives:

$$\frac{N\lambda^2}{A} = \exp\left(-\frac{\epsilon}{k_B T}\right)\frac{N_s\lambda^2}{A_s}$$

Which implies:

$$N_s = N\frac{A_s}{A}\exp\left(\frac{\epsilon}{k_B T}\right)$$

A system has a grand canonical partition function:

$$\Theta = \exp\left[\beta\mu + (\beta\mu)^2\right]$$

What is the average energy $\langle E\rangle$ of the system?

Select one alternative:

- $\langle E\rangle = \beta\mu^2$
- $\langle E\rangle = \mu$
- $\langle E\rangle = \frac{1}{\beta} + 3\mu$
- $\langle E\rangle = -\mu - \beta\mu^2$
- $\langle E\rangle = 0$
- $\langle E\rangle = \frac{1}{\beta}$

Solution 24

Average particle number:

$$\langle N\rangle = \frac{\partial}{\beta\partial\mu}\ln\Theta = \frac{\partial}{\beta\partial\mu}\left[\beta\mu + (\beta\mu)^2\right] = 1 + 2\beta\mu$$

Average energy:

$$\begin{aligned}\langle E\rangle &= \mu\langle N\rangle - \frac{\partial}{\partial\beta}\ln\Theta \\ &= \mu(1+2\beta\mu) - (\mu + 2\mu^2\beta) \\ &= 0\end{aligned}$$

A system has grand partition function:
$\Theta = \exp(M\beta\mu - \beta\epsilon)$
where $\epsilon > 0$ is a constant, and $M \gg 1$ a constant number. μ is the chemical potential and $\beta = \frac{1}{k_B T}$. The fluctuations of the particle number $\Delta N^2 = \langle N^2\rangle - \langle N\rangle^2$ in the system is:

Select one alternative:

- $\Delta N^2 = \infty$
- $\Delta N^2 = 0$
- $\Delta N^2 = M\exp\left(\frac{\epsilon}{\mu}\right)$
- $\Delta N^2 = M\frac{\epsilon}{\mu}$
- $\Delta N^2 = M$
- $\Delta N^2 = M^2$

Solution 25 Mean value :

$$\begin{aligned}\langle N\rangle &= \frac{\partial}{\beta\partial\mu}\ln\Theta \\ &= \frac{\partial}{\beta\partial\mu}\left[M\beta\mu - \beta\epsilon\right] \\ &= M\end{aligned}$$

Fluctuations around mean:

$$\Delta N^2 = \frac{\partial}{\beta\partial\mu}\langle N\rangle = 0$$

A system has canonical partition function Z_N, where N is the number of particles. The system cannot have more than N_c particles.
When $N \leq N_c$ the partition function is $Z_N = \mathrm{e}^{\beta\epsilon N}$, and when $N > N_c$ the partition function is $Z_N = 0$. $\epsilon > 0$ is a constant. The system is connected to a particle reservoir with chemical potential $\mu = 0$.
When $\beta\epsilon \gg 1$ the grand partition function is:

Select one alternative:

- $\Theta \approx N_c \exp(N_c\, \beta\epsilon)$
- $\Theta \approx \exp(\beta\epsilon)$
- $\Theta \approx 1$
- $\Theta \approx N_c \exp(\beta\epsilon)$
- $\Theta \approx \exp(\exp(N_c\, \beta\epsilon))$
- $\Theta \approx \exp(N_c\, \beta\epsilon)$

Solution 26

$$\begin{aligned}
\Theta &= \sum_{N=0}^{N_c} Z_N \\
&= \sum_{N=0}^{N_c} \mathrm{e}^{\beta\epsilon N} \\
&= \frac{\mathrm{e}^{\beta\epsilon(N_c+1)} - 1}{\mathrm{e}^{\beta\epsilon} - 1}
\end{aligned}$$

If $\beta\epsilon \gg 1$:

$$\Theta \approx \mathrm{e}^{\beta\epsilon N_c}$$

A system has canonical partition function $Z_N = KV^{N/3}(k_B T)^{N/5}$, where K is a constant, N the number of particles and V the volume. The pressure p and heat capacity C of the system are:

Select one alternative:

- $p = 0$ and $C = \frac{N}{3}k_B$
- $p = \frac{N}{V}k_B T$ and $C = \frac{3N}{2}k_B$
- $p = \frac{N}{3V}k_B T$ and $C = \frac{N}{5}k_B$
- $p = \frac{N}{3V}k_B T$ and $C = 0$
- $p = \frac{1}{3V}k_B T$ and $C = \frac{1}{5}k_B$
- $p = \frac{3N}{5V}k_B T$ and $C = \frac{5N}{3}k_B$

Solution 27

Free energy:

$$\begin{aligned} F &= -\frac{1}{\beta}\ln(Z) \\ &= -\frac{1}{\beta}\ln(K) - \frac{1}{\beta}\frac{N}{3}\ln(V) + \frac{N}{5}\ln(\beta) \end{aligned}$$

Pressure:

$$p = -\frac{\partial F}{\partial V} = \frac{N}{3V}k_B T$$

Mean energy:

$$\langle E \rangle = -\frac{\partial}{\partial \beta}\ln Z = \frac{N}{5}k_B T$$

Heat capacity:

$$C = \frac{\partial}{\partial T}\langle E \rangle = \frac{N}{5}k_B$$

A box of volume V contains an ideal classical gas of N molecules, with chemical potential $\mu_g = k_B T \ln(N\lambda^3/V)$, λ is the thermal de Broglie length. We introduce two spherical objects (beads) inside the box, and the molecules can adsorb on these objects. The objects are far apart (not touching). The chemical potential of adsorption on sphere 1 is:

$$\mu_1 = -\epsilon_1 + k_B T \ln(c N_1)$$

and similarly for sphere 2:

$$\mu_2 = -\epsilon_2 + k_B T \ln(c N_2)$$

where N_1 and N_2 are the number of adsorbed molecules on sphere 1 and 2, and ϵ_1, ϵ_2 c are constants. We always have $N_1 \ll N$ and $N_2 \ll N$. In thermal equilibrium the ratio $\frac{N_1}{N_2}$ is:

Select one alternative:

- $\frac{N_1}{N_2} = \exp\left[-\frac{\epsilon_1}{k_B T}\right] + \exp\left[-\frac{\epsilon_2}{k_B T}\right]$
- $\frac{N_1}{N_2} = \exp\left[\frac{(\epsilon_1 - \epsilon_2)}{k_B T}\right]$
- $\frac{N_1}{N_2} = \frac{\epsilon_1}{\epsilon_2}\exp\left[-\frac{\epsilon_1}{k_B T}\right] + \frac{\epsilon_2}{\epsilon_1}\exp\left[-\frac{\epsilon_2}{k_B T}\right]$
- $\frac{N_1}{N_2} = \frac{\epsilon_1}{\epsilon_2}\exp\left[\frac{(\epsilon_1 - \epsilon_2)}{k_B T}\right]$
- $\frac{N_1}{N_2} = 1$
- $\frac{N_1}{N_2} = \frac{\epsilon_1}{\epsilon_2}$

Solution 28

The gas acts as a particle reservoir for adsorption on the two beads. In thermal equilibrium we must have

$$\mu_1 = \mu_g$$

and

$$\mu_2 = \mu_g$$

Which implies $\mu_1 = \mu_2$ and :

$$-\epsilon_1 + k_B T \ln cN_1 = -\epsilon_2 + k_B T \ln cN_2$$

Which implies:

$$\frac{N_1}{N_2} = \exp\left[\frac{\epsilon_1 - \epsilon_2}{k_B T}\right]$$

Consider the 1D Ising model with an external magnetic field:

$$H = -J\sum_{i=1}^{N-1} s_i s_{i+1} - \mu B \sum_{i=1}^{N} s_i$$

where μ is the magnetic moment. We are in the canonical ensemble.
The heat capacity is $C = \frac{\partial}{\partial T}\langle H\rangle$. The heat capacity at zero and infinite temperature is denoted by $C_0 = C(T=0)$ and $C_\infty = C(T\to\infty)$.
Which one of these statements is correct?

Select one alternative:

- $C_0 = 0$ and $C_\infty = Nk_B$
- $C_0 = 0$ and $C_\infty = k_B\frac{\mu B}{J}$
- $C_0 = Nk_B$ and $C_\infty = 0$
- $C_0 = 0$ and $C_\infty = 0$
- $C_0 = 0$ and $C_\infty = \infty$
- $C_0 = Nk_B$ and $C_\infty = Nk_B$

Solution 29

At low temperatures the spins in the Ising model are all parallel and locked in that configuration. There is not enough thermal energy to flip a spin and the heat capacity is therefore zero $C_0 = 0$. At high temperatures the spin directions are random and the mean energy tends towards zero $\langle E\rangle = 0$, the mean energy does not change if we increase the temperature, therefore the heat capacity is zero: $C_\infty = 0$.

A polymer in two dimensions has 5 monomers, numbered $n = 0, 1, 2, 3, 4$. The position of monomer n is $\vec{R}_n = \sum_{i=1}^{n} \vec{t}_i$, where $\vec{t}_i$ are the bond vectors.

The angles θ_i between two consecutive bond vectors ($\vec{t}_i \cdot \vec{t}_{i+1} = \cos(\theta_i)$) can only take three values: $\theta_i = 0, \alpha, -\alpha$

The deformation energy of the polymer is $H = -\epsilon(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$.
The canonical partition function $Z = \sum_{\theta_1,\theta_2,\theta_3} \exp(-\beta H)$ of the polymer is:

Select one alternative:

- $Z = \exp(\beta\epsilon) + 2\exp[\beta\epsilon\cos(\alpha)]$
- $Z = Z_1^3$ where $Z_1 = 1 + 2\exp[\beta\epsilon]$
- $Z = Z_1^4$ where $Z_1 = 1 + 2\exp[\beta\epsilon\cos(\alpha)]$
- $Z = Z_1^3$ where $Z_1 = 1 + 2\exp[\beta\epsilon\cos(\alpha)]$
- $Z = Z_1^2$ where $Z_1 = 1 + 2\exp[\beta\epsilon]$
- $Z = Z_1^3$ where $Z_1 = \exp(\beta\epsilon) + 2\exp[\beta\epsilon\cos(\alpha)]$

Solution 30

Summing over all configurations results in the partition function:

$$\begin{aligned} Z &= \sum_{\theta_1,\theta_2,\theta_3} \mathrm{e}^{\beta\epsilon(\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3))} \\ &= \sum_{\theta_1} \mathrm{e}^{\beta\epsilon\cos(\theta_1)} \sum_{\theta_2} \mathrm{e}^{\beta\epsilon\cos(\theta_2)} \sum_{\theta_3} \mathrm{e}^{\beta\epsilon\cos(\theta_3)} \\ &= \left[\mathrm{e}^{\beta\epsilon} + \mathrm{e}^{\beta\epsilon\cos(\alpha)} + \mathrm{e}^{\beta\epsilon\cos(-\alpha)}\right]^3 \\ &= \left[\mathrm{e}^{\beta\epsilon} + 2\mathrm{e}^{\beta\epsilon\cos(\alpha)}\right]^3 \end{aligned}$$

In the Debye model for heat capacity of solids it is assumed that the density of states is quadratic $g(\omega) \propto \omega^2$ up to the cut-off frequency ω_D. This gives a heat capacity at low temperature that is proportional to temperature to the power three: $C \propto T^3$. What would be the low temperature behaviour if we instead assumed that the density of states is linear: $g(\omega) \propto \omega$ up to the cut-off frequency.

Select one alternative:

- $C(T) \sim T^4$
- $C(T) \sim T^3$
- $C(T) \sim T^{3/2}$
- $C(T) \sim T$
- $C(T) \sim T^2$
- $C(T) \sim T^{5/2}$

Solution 31

Mean energy:

$$\langle E \rangle = \int_0^{\omega_D} \frac{g(\omega)\omega \mathrm{d}\omega}{\mathrm{e}^{\beta\hbar\omega} - 1}$$

If $g(\omega) \propto \omega$ we get

$$\langle E \rangle \propto \int_0^{\omega_D} \frac{\omega^2 \mathrm{d}\omega}{\mathrm{e}^{\beta\hbar\omega} - 1} = \frac{1}{(\beta\hbar)^3} \int_0^{\beta\hbar\omega_D} \frac{x^2 dx}{\mathrm{e}^x - 1}$$

Which implies $\langle E \rangle \propto 1/\beta^3 \sim T^3$ at low temperatures.
The heat capacity therefor scales as $C \sim T^2$.

Consider a system with Hamiltonian $H_N = -h(s_1 + s_2 + \cdots + s_N)$,
where $h > 0$, $s_i = \pm 1$ and N is the number of particles in the system.
The canonical partition function is given by: $Z_N = \sum_{\{s_i\}} \mathrm{e}^{-\beta H_N}$.

We are in the grand canonical ensemble and the chemical potential is μ. The grand partition function is a convergent series when $N \to \infty$ provided $\mu < 0$ is sufficiently small, in which case Θ becomes:

Select one alternative:

- $\Theta = \frac{1}{1+\mathrm{e}^{2\beta\mu}(\mathrm{e}^{2\beta h}+\mathrm{e}^{-2\beta h})}$
- $\Theta = \frac{1}{\mathrm{e}^{\beta\mu}+\mathrm{e}^{\beta h}+\mathrm{e}^{-\beta h}}$
- $\Theta = \frac{1}{1-\mathrm{e}^{\beta\mu}(\mathrm{e}^{\beta h}+\mathrm{e}^{-\beta h})}$
- $\Theta = \frac{1}{1-\mathrm{e}^{\beta\mu}}$
- $\Theta = \frac{1}{1-\mathrm{e}^{\beta h}-\mathrm{e}^{-\beta h}}$
- $\Theta = \frac{1}{1-\mathrm{e}^{\beta h}-\mathrm{e}^{-\beta h}-\mathrm{e}^{\beta\mu}}$

Solution 32 Canonical partition function:

$$
\begin{aligned}
Z_N &= \sum_{s_1, s_2, \cdots, s_N} \mathrm{e}^{-\beta H_N} \\
&= \sum_{s_1} \mathrm{e}^{\beta h s_1} \sum_{s_2} \mathrm{e}^{\beta h s_2} \cdots \sum_{s_N} \mathrm{e}^{\beta h s_N} \\
&= \left(\mathrm{e}^{\beta h} + \mathrm{e}^{-\beta h}\right)^N
\end{aligned}
$$

Grand partition function:

$$
\begin{aligned}
\Theta &= \sum_{N=0}^{\infty} \mathrm{e}^{\beta\mu N} Z_N \\
&= \sum_{N=0}^{\infty} \mathrm{e}^{\beta\mu N} \left(\mathrm{e}^{\beta h} + \mathrm{e}^{-\beta h}\right)^N
\end{aligned}
$$

This geometric series converges if $\mathrm{e}^{\beta\mu}\left(\mathrm{e}^{\beta h} + \mathrm{e}^{-\beta h}\right) < 1$, in which case we get:

$$
\Theta = \frac{1}{1 - \mathrm{e}^{\beta\mu}\left(\mathrm{e}^{\beta h} + \mathrm{e}^{-\beta h}\right)}
$$

The grand partition function of a classical ideal gas is:

$$\Theta = \exp\left[\frac{V}{\lambda^3}\exp\left(\beta\mu\right)\right]$$

where λ is the thermal de Broglie length. The density of the gas is $\rho = \frac{\langle N\rangle}{V}$. What happens to the density if we keep μ constant and let $T \to \infty$?

Select one alternative:

- Approaches zero following high temperature behaviour $\rho \propto 1/\sqrt{T}$
- Approaches a constant density $\rho = \text{constant}$
- Approaches zero following high temperature behaviour $\rho \propto 1/T^2$
- Approaches zero following high temperature behaviour $\rho \propto 1/T^4$
- Diverges following the high temperature behaviour $\rho \propto T^{3/2}$
- Diverges following the high temperature behaviour $\rho \propto T^3$

Solution 33

Mean number of particles:

$$\begin{aligned}\langle N\rangle &= \frac{\partial}{\beta\partial\mu}\ln\Theta \\ &= \frac{\partial}{\beta\partial\mu}\frac{V}{\lambda^3}\mathrm{e}^{\beta\mu} \\ &= \frac{V}{\lambda^3}\mathrm{e}^{\beta\mu}\end{aligned}$$

At high temperatures $\beta\mu$ is small and we get

$$\rho = \frac{\langle N\rangle}{V} \approx \frac{1}{\lambda^3}$$

Thermal de Broglie length:

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} \propto T^{-1/2}$$

When μ is kept constant ρ therefore diverges at high temperatures:

$$\rho \propto T^{3/2}$$

Consider a particle living on the z-axis, and subject to a gravitational field. The Hamiltonian is: $H = \frac{p_z^2}{2m} + mgz$. The particle is confined to the interval $0 \leq z \leq L$. We can think of this as a particle in a one-dimensional box. We are in the canonical ensemble. The pressure at $z = L$ is defined as $P = -\frac{\partial F}{\partial L}$ where $F = -\frac{1}{\beta}\ln Z$ is the free energy of the particle. Calculate F and find P. Which of these is the correct expression for the pressure P?

Select one alternative:

- $P = \frac{mg}{L}$
- $P = \frac{k_B T}{L^2}$
- $P = \frac{mg}{e^{\beta mgL} - 1}$
- $P = \frac{k_B T}{2L}$
- $P = \frac{mg}{L}$
- $P = \frac{k_B T}{e^{\beta mgL} + 1}$

Solution 34

Partition function:

$$
\begin{aligned}
Z &= \frac{1}{h}\int_{-\infty}^{\infty} \mathrm{d}p_x \mathrm{e}^{-\beta\frac{p_x^2}{2m}} \int_0^L \mathrm{d}z \mathrm{e}^{-\beta mgz} \\
&= \frac{1}{h}\left(\frac{2\pi m}{\beta}\right)^{1/2} \frac{1}{mg\beta}\left(1 - \mathrm{e}^{-\beta mgL}\right)
\end{aligned}
$$

Pressure:

$$
\begin{aligned}
p &= -\frac{\partial}{\partial L} F \\
&= \frac{1}{\beta}\frac{\partial}{\partial L} \ln Z \\
&= \frac{1}{\beta}\frac{\partial}{\partial L} \ln\left(1 - \mathrm{e}^{-\beta mgL}\right) \\
&= \frac{mg\,\mathrm{e}^{-\beta mgL}}{1 - \mathrm{e}^{-\beta mgL}} \\
&= \frac{mg}{\mathrm{e}^{\beta mgL} - 1}
\end{aligned}
$$

A system has the possible energy levels $E = n\epsilon$ where $n = 0, 1, 2, 3, \cdots, N$, and $\epsilon > 0$ a constant. For a given energy level n there are $\Gamma_n = \frac{N!}{(N-n)!\,n!}$ different configurations of the system. We are in the microcanonical ensemble. Calculate the entropy $S = k_B \ln \Gamma_n$ using Stirling's formula, and assuming that N, $N-n$ and n are all large numbers.
The relation $\frac{dS}{dE} = \frac{1}{T}$ implies:

Select one alternative:

- $\frac{n}{N-n} = \mathrm{e}^{-\frac{\epsilon}{k_B T}}$
- $\frac{n}{N-n} = \frac{k_B T}{\epsilon}$
- $\frac{n}{2N} = \mathrm{e}^{-\frac{2\epsilon}{k_B T}}$
- $\frac{n}{N} = \frac{\epsilon}{k_B T}$
- $\frac{n}{N} = \frac{k_B T}{\epsilon}$
- $n = \mathrm{e}^{\frac{\epsilon}{k_B T}}$

Solution 35

Using Stirlings formula the entropy can be written as:

$$S = k_B \ln(\Gamma_n) \approx k_B \ln(N!) - k_B n \ln(n) + k_B n - k_B (N-n) \ln(N-n) + k_B \ln(N-n)$$

Which implies:

$$\frac{dS}{dE} = \frac{1}{\epsilon}\frac{dS}{dn} = \frac{k_B}{\epsilon} \ln\left(\frac{N-n}{n}\right)$$

Hence:

$$\frac{n}{N-n} = \mathrm{e}^{-\frac{\epsilon}{k_B T}}$$

Consider a system with three energy levels $\epsilon_0 = 0$, $\epsilon_1 = \epsilon$, and $\epsilon_2 = 2\epsilon$. The system is populated with $N = 2$ bosons following Bose-Einstein statistics. The canonical partition function is:

Select one alternative:

- $Z = 1 + \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-6\beta\epsilon}$
- $Z = 1 + \mathrm{e}^{-\beta\epsilon} + 2\mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon}$
- $Z = \mathrm{e}^{-\beta\epsilon} + \mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon}$
- $Z = 1 + \mathrm{e}^{-\beta\epsilon} + \mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-3\beta\epsilon}$
- $Z = 1 + \mathrm{e}^{-4\beta\epsilon}$
- $Z = 1 + \mathrm{e}^{-\beta\epsilon} + 2\mathrm{e}^{-2\beta\epsilon} + 4\mathrm{e}^{-3\beta\epsilon} + 8\mathrm{e}^{-4\beta\epsilon}$

Solution 36

The possible occupation numbers of energy levels and corresponding total energy:

(n_0, n_1, n_2)	energy
$(1,1,0)$	ϵ
$(1,0,1)$	2ϵ
$(0,1,1)$	3ϵ
$(2,0,0)$	0
$(0,2,0)$	2ϵ
$(0,0,2)$	4ϵ

$$Z = 1 + \mathrm{e}^{-\beta\epsilon} + 2\mathrm{e}^{-2\beta\epsilon} + \mathrm{e}^{-3\beta\epsilon} + \mathrm{e}^{-4\beta\epsilon}$$

A system with three energy levels $\epsilon_1 = 0$,$\epsilon_1 = \epsilon$, and $\epsilon_1 = 2\epsilon$ is populated by $N = 2$ indistinguishable fermions obeying Fermi-Dirac statistics. The canonical partition function is:

Select one alternative:

- $Z = 1 + e^{3\beta\epsilon}$
- $Z = e^{\beta\epsilon} + e^{3\beta\epsilon}$
- $Z = e^{-\beta\epsilon} + e^{-2\beta\epsilon}$
- $Z = e^{-\beta\epsilon} + e^{-2\beta\epsilon} + e^{-3\beta\epsilon}$
- $Z = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}$
- $Z = e^{-2\beta\epsilon} + e^{-4\beta\epsilon}$

Solution 37 The possible occupation numbers of energy levels and corresponding total energy:

(n_0, n_1, n_2)	energy
$(1, 1, 0)$	ϵ
$(1, 0, 1)$	2ϵ
$(0, 1, 1)$	3ϵ

$$Z = e^{-\beta\epsilon} + e^{-2\beta\epsilon} + e^{-3\beta\epsilon}$$

Consider a system of bosons in one dimension inside a harmonic potential.
The energy is $\epsilon = (n + \frac{1}{2})\hbar\omega$, where $n = 0, 1, 2, \cdots, \infty$. We are in the grand canonical ensemble.
Find the density of states $g(\epsilon)$. In the continuum limit the average number of particles in the excited states (not ground state) is:

Select one alternative:

- $\langle N_{ex} \rangle = \frac{1}{(\hbar\omega)^2} \int_0^\infty \frac{\mathrm{d}\epsilon\, \epsilon}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}$
- $\langle N_{ex} \rangle = \frac{1}{(\hbar\omega)^3} \int_0^\infty \frac{\mathrm{d}\epsilon\, \epsilon}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}$
- $\langle N_{ex} \rangle = \frac{1}{\hbar\omega} \int_0^\infty \frac{\mathrm{d}\epsilon}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}$
- $\langle N_{ex} \rangle = \int_0^\infty \frac{\mathrm{d}\epsilon}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}$
- $\langle N_{ex} \rangle = \int_0^\infty \frac{\mathrm{d}\epsilon\, \hbar\omega^2}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}$
- $\langle N_{ex} \rangle = \frac{1}{\hbar\omega} \int_0^\infty \frac{\mathrm{d}\epsilon\, \epsilon^2}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}$

Solution 38

From the expression for the energy we get $\mathrm{d}\epsilon = \hbar\omega \mathrm{d}n$ which implies that the density of states is $g(\omega) = \frac{1}{\hbar\omega}$.

The number of excited states is therefore

$$\langle N_{ex} =\rangle \frac{1}{\hbar\omega} \int_0^\infty \frac{\mathrm{d}\epsilon}{\mathrm{e}^{\beta(\mu-\epsilon)} - 1}$$

A system has three energy levels, a ground state $\epsilon_1 = 0$ and two excited states with the same energy $\epsilon_2 = \epsilon$, $\epsilon_3 = \epsilon$. The system is populated with bosons obeying Bose-Einstein statistics. We are in the grand canonical ensemble. The average number of particles in the system is $\langle N \rangle$. On average half of the particles are in the ground state, what is the temperature?

Select one alternative:

- $k_B T = \frac{\epsilon}{\ln\left(\frac{N+4}{N+2}\right)}$
- $k_B T = \frac{\epsilon}{\ln(2)}$
- $k_B T = 2\epsilon$
- $k_B T = \epsilon$
- $k_B T = \frac{\epsilon}{2N}$
- $k_B T = \frac{\epsilon}{\ln(N)}$

Solution 39

Total number of particles:

$$\begin{aligned}
\langle N \rangle &= \langle n_0 \rangle + \langle n_1 \rangle + \langle n_2 \rangle \\
&= \frac{1}{\mathrm{e}^{-\beta\mu} - 1} + \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1} + \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1} \\
&= \frac{1}{\mathrm{e}^{-\beta\mu} - 1} + \frac{2}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1}
\end{aligned}$$

Half of the particles are in the ground state:

$$\langle n_0 \rangle = \frac{N}{2} \quad \Rightarrow \quad \mathrm{e}^{-\beta\mu} = 1 + \frac{2}{N}$$

Half of the particles are in the excited states:

$$\langle n_1 \rangle + \langle n_1 \rangle = \frac{N}{2} \quad \Rightarrow \quad \mathrm{e}^{\beta(\epsilon-\mu)} = 1 + \frac{4}{N}$$

This implies :

$$\mathrm{e}^{\beta\epsilon} = \frac{1 + \frac{4}{N}}{1 + \frac{2}{N}} \quad \Rightarrow \quad k_B T = \frac{\epsilon}{\ln\left(\frac{N+4}{N+2}\right)}$$

Consider a single gas particle with kinetic energy $E = \frac{p_x^2+p_y^2+p_z^2}{2m}$ confined inside a volume V. We are in the canonical ensemble. What is the fluctuations in energy of the particle $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$?

Select one alternative:

- $\Delta E^2 = \frac{1}{4}(k_B T)^2$
- $\Delta E^2 = (k_B T)^2$
- $\Delta E^2 = 2(k_B T)^2$
- $\Delta E^2 = \frac{3}{5}(k_B T)$
- $\Delta E^2 = 4(k_B T)^2$
- $\Delta E^2 = \frac{3}{2}(k_B T)^2$

Solution 40

$$\langle E \rangle = \left\langle \frac{p_x^2 + p_y^2 + p_z^2}{2m} \right\rangle = \frac{3}{2} k_B T = \frac{3}{2\beta}$$

$$\Delta E^2 = -\frac{\partial}{\partial \beta} \langle E \rangle = -\frac{\partial}{\partial \beta} \frac{3}{2\beta} = \frac{3}{2}(k_B T)^2$$