NTNU **Institutt for fysikk**

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Exam in TFY4240 Electromagnetic Theory

Dec 14, 2010
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 5 pages.

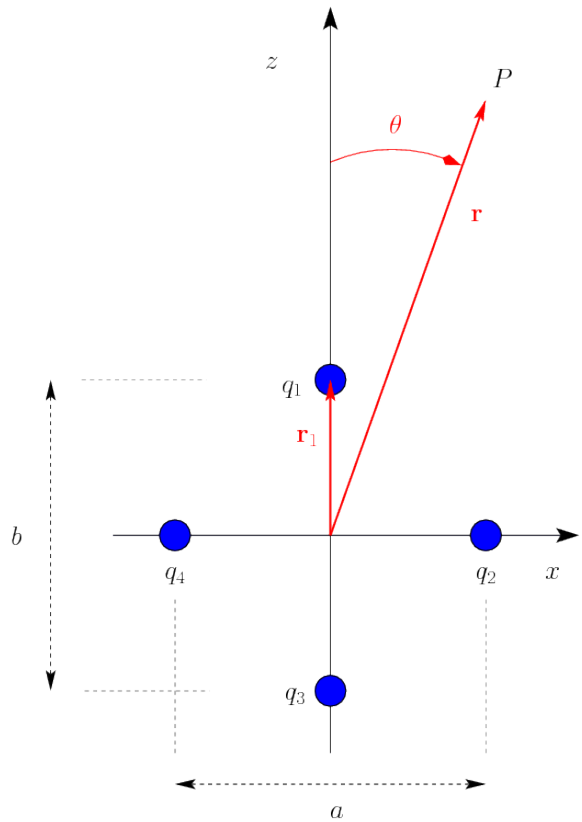
This exam consists of two problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading, except point 2e that will be given double weight. For your information, I estimate that you will spend about twice the amount of time on the 2nd problem relative the 1st.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do around 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Good luck to all of you!

Problem 1.



Consider a rectangle of horizontal and vertical diagonals a and b, respectively, and where we have placed (static) charges q_i $(i = 1, \ldots, 4)$ at each corner as shown in the above figure.
A coordinate system is placed with its origin at the center of this rectangle so that the rectangle is in the xz-plane. Relative to this coordinate system, the distance vector to the observation point P will be denoted $\mathbf{r}$, and the distance vector to charge i is $\mathbf{r}_i$.

a) Write down the general expression for the (total) scalar potential $V(\mathbf{r})$ for the four charges that is valid at *any* point $\mathbf{r}$ $(\neq \mathbf{r}_i)$.

b) Describe, in your own words, what is meant by a multi-pole expansion for the scalar potential. In particular, point out the essential assumption that must be satisfied for the first few terms of this expansion to represent a good approximation to the potential.

We will from now onwards assume that the dimensions of the rectangle are *small* compared to the distance, $\mathbf{r}$, to the observation point P; that is $a/r \ll 1$ and $b/r \ll 1$ with $|\mathbf{r}| = r$. Approximations to the scalar potential will now be studied for various charge configurations when the observation point is far away from all the charges. With an approximation, we mean the dominating ("the largest") non-zero term contributing to the potential.

c) Write down an approximative expression for the potential $V(\mathbf{r})$ (valid for large r) for the four charges under the assumption that $q_1 + q_2 + q_3 + q_4 \neq 0$.

d) Now set $q_2 = q_4 = 0$, $q_1 = -q_3 = q$ and $b = \ell > 0$. Show in this case that the potential for the system can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \tag{1}$$

and give an expression for $\mathbf{p}$. What is the quantity $\mathbf{p}$ called? Here $\hat{\mathbf{r}}$ denotes a unit vector in the radial direction. Make a sketch the angular distribution of the of this potential (for given r).

Assume now that all charges are non-zero ($q_i \neq 0$ for all i); $b = \ell > 0$ (with $\ell/r \ll 1$); and that $a/b = a/\ell \ll 1$.

e) We first consider the case where $q_1 = q_4 = q$ and $q_2 = q_3 = -q$. Make a sketch of the charge distribution indicating the positions of the positive and negative charges. What is the potential $V(\mathbf{r})$ in this case? [*Hint*: Make use of the physics of the problem, and do not try to calculate this directly Of course, such a more lengthy calculation will work though]. Make a sketch of the of the angular distribution of the potential (for a given distance r).

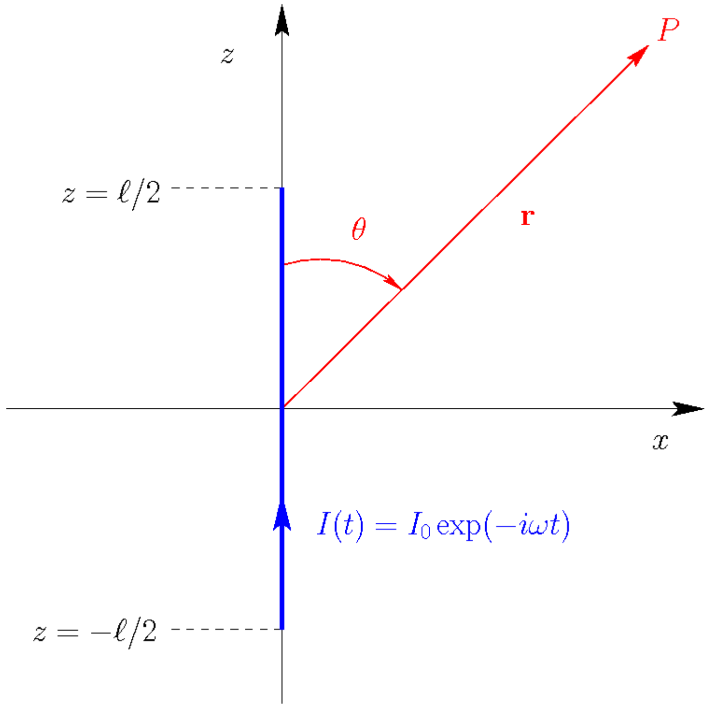
f) Assume now instead that $q_1 = q_3 = q$ and $q_2 = q_4 = -q$. Also for this case make a sketch of the charge distribution indicating the positions of the positive and negative charges. Why will in this case (the angular distribution of) the potential be more complicated than in the previous sub-problem?

g) Find an approximative expression for the potential $V(\mathbf{r})$ in the above case (*i.e.* when $q_1 = q_3 = q$ and $q_2 = q_4 = -q$) and show that it satisfies

$$V(\mathbf{r}) \propto \frac{1}{r^3}. \tag{2}$$

Make a sketch of the angular distribution of the potential in this case (assuming a constant distance r). What is this angular pattern called?

Problem 2.



We consider a thin, straight, conducing wire of length ℓ that is oriented along the z-axis as shown in the figure above. The wire carries the time-varying current

$$I(t) = I_0 \exp\left(-i\omega t\right), \tag{3}$$

everywhere along its length ℓ. Here I_0 and ω are both positive constants. The system is a simple model for an antenna.

a) Charge will only build up at the endpoints of the wire. Explain why this is so. Find an expression for the time-dependent charge, $Q(t)$, building up at one endpoint.

b) Use your expression for $Q(t)$ to determine the dipole moment $\mathbf{p}(t)$ of the simple antenna.

c) Argue why the current density can be written as

$$\mathbf{J}(\mathbf{r}, t) = \hat{\mathbf{z}}\, I(t)\delta(x)\delta(y)\theta(\ell/2) - |z|), \tag{4}$$

where $\delta(\cdot)$ is the Dirac delta-function, $\theta(\cdot)$ is the (Heaviside) step-function, and $\hat{\mathbf{z}}$ is a unit vector in the positive z-direction.

We will now study the electromagnetic field radiated from the antenna. To this end, we will start by calculate the potentials. In the Lorentz-gauge the vector potential satisfies the equation

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2}\partial_t^2 \mathbf{A}(\mathbf{r}, t) = -\mu_0 \mathbf{J}(\mathbf{r}, t), \tag{5}$$

where $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in vacuum. Here ε_0 and μ_0 are the electric permitivity and magnetic permeability of free space , respectively.

d) By using the Green's function for the wave-equation

$$g(\mathbf{r},t|\mathbf{r}',t') = -\frac{\delta\left(t - t' - \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{6}$$

show that a solution to Eq. (5) is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}. \tag{7}$$

What is the meaning of $\mathbf{r}$ and $\mathbf{r}'$? Moreover, what is the expression for t_r, and what is the physical interpretation of this quantity?

e) Calculate the vector potential $\mathbf{A}(\mathbf{r},t)$ under the assumption that $r \gg \ell$ and show that it can be written as $(k = \omega/c)$

$$\mathbf{A}(\mathbf{r},t) \approx \hat{\mathbf{z}}\,\frac{\mu_0}{4\pi}\frac{2I_0}{\cos\theta}\sin\left(\frac{k\ell}{2}\cos\theta\right)\frac{\exp\left(ikr - i\omega t\right)}{kr}. \tag{8}$$

It will now be assumed that the wavelength ($\lambda = 2\pi/k$) and distance to the observation point (r) are so that $kr \gg 1$.

f) Derive an expression for the magnetic field $\mathbf{H}(\mathbf{r},t)$ (under the condition $kr \gg 1$). [*Answer* : $\mathbf{H} \approx \frac{1}{\mu_0}\, i\mathbf{k}\times\mathbf{A}$.]

g) Calculate the corresponding electric field $\mathbf{E}(\mathbf{r},t)$ (under the same assumption as in the previous sub-problem). [*Hint* : Use Amperes law].

h) Obtain an expression for the time-averaged Poyntings vector, $\langle\mathbf{S}\rangle_t$ $(= \frac{1}{2}\mathbf{E}\times\mathbf{H}^*)$, in terms of the known quantities of the problem.

The radiation pattern of the antenna is defined as

$$\frac{dP}{d\Omega} = |\langle\mathbf{S}\rangle_t|\, r^2, \tag{9}$$

where $P = \int d\Omega\, dP/d\Omega = \int \langle\mathbf{S}\rangle_t \cdot d\mathbf{A}$ is the total power radiated by the antenna.

i) Calculate $dP/d\Omega$ and show that

$$\frac{dP}{d\Omega} \propto \sin^2\left(\frac{k\ell}{2}\cos\theta\right)\tan^2\theta. \tag{10}$$

j) Assume that the antenna is small compared to the wavelength, *i.e.* $k\ell \ll 1$, and obtain the expression for $dP/d\Omega$ in this limit.

k) Argue why your expression for $dP/d\Omega$ in the limit $k\ell \ll 1$ is reasonable. Make a sketch of the radiation pattern in this case. What is this pattern called?

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$