NTNU **Institutt for fysikk**

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Exam in TFY4240 Electromagnetic Theory

Dec 20, 2013
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 6 pages.

This exam consists of three problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading. However, some sub-problems may be given double weight, but only if so is indicated explicitly.

For your information, it is estimated that you will spend about 10% (or less) of the available time on problem 1; 50% on problem 2; and about 40% on problem 3.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do around 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Note that some formulas are given after the last problem!

Good luck to all of you!

Problem 1.

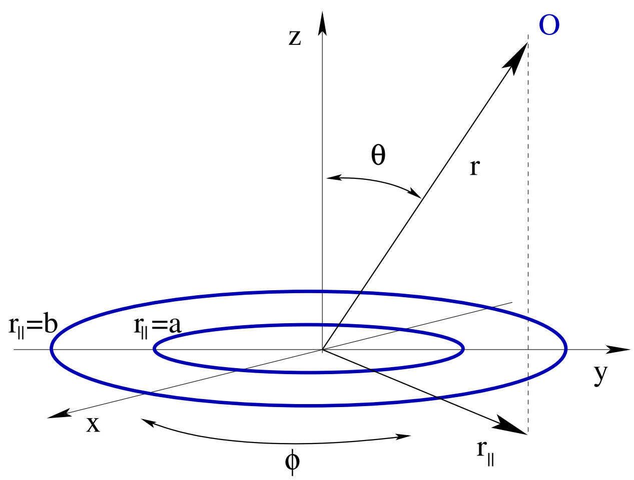

Figure 1: Schematics for problem 1

A very long non-magnetic cylindrical conductor with inner radius a and outer radius b carries a constant current I. The current density in the conductor is uniform. A coordinate system is defined so that the cylindrical axis coincides with the z-axis, and the direction of the current is along $\hat{\boldsymbol{z}}$ as depicted in Fig. 1.

a) Obtain an expression for the current density, $\boldsymbol{J}(\boldsymbol{r})$, of the system (where $\boldsymbol{r}$ is a general position vector).

b) Obtain the *magnetic field*, $\boldsymbol{H}(\boldsymbol{r})$, as function of the distance from the z-axis, $r_\parallel = |\boldsymbol{r}_\parallel|$ ($\boldsymbol{r}_\parallel = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$), for the regions: (i) $r_\parallel < a$; (ii) $a < r_\parallel < b$; and (iii) $r_\parallel > b$. Make a sketch of the amplitude of the magnetic field. Remember to indicate the direction of the magnetic field.

c) Repeat the previous sub-problem, but now for the *electric field*. [1]

Problem 2.

A dielectric sphere of radius a and dielectric constant ε_1 is placed in a dielectric liquid of infinite extent and dielectric constant ε_2 (see Fig. 2). Originally a uniform static electric field, $\boldsymbol{E}_0$, is present in the liquid, *i.e.*, before the sphere is placed in the liquid. We choose a coordinate system with the origin located at the center of the sphere and the positive z-axis directed along the original electric field $\boldsymbol{E}_0 = E_0\hat{\boldsymbol{z}}$ ($E_0 > 0$), as illustrated in Fig. 2. As usual, $\hat{\boldsymbol{z}}$ denotes the unit vector in the z-direction.

The purpose of this problem is to find the scalar potential and the total electric field inside and outside the dielectric sphere so we can determine how it reacts to the external field.

a) Argue why the scalar potential, $V(\boldsymbol{r})$, for the system can be written in the form

$$V(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{1}$$

[1] This means: Obtain the *electric field*, $\boldsymbol{E}(\boldsymbol{r})$, as function of $r_\parallel = |\boldsymbol{r}_\parallel|$ for the regions: (i) $r_\parallel < a$; (ii) $a < r_\parallel < b$; and (iii) $r_\parallel > b$.

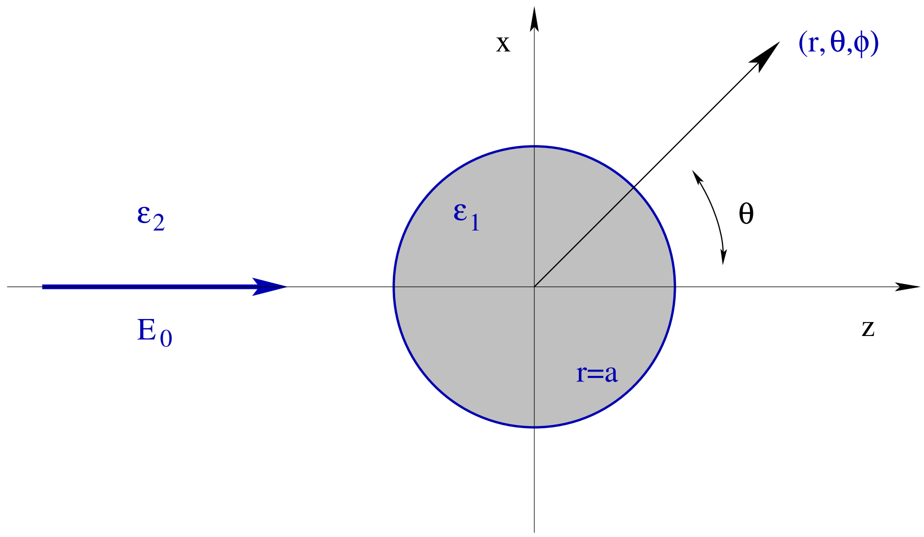

Figure 2: Schematics for problem 2

Explain why we have both positive and negative powers of r, and what the functions $P_\ell(\cos\theta)$ are. *Note* that you are not asked to derive this result, only motivate it!

b) Write down the *four* boundary conditions satisfied by the scalar potential at (i) $r = 0$; (ii) $r \to \infty$; and (iii) $r = a$. [Given $P_1(x) = x$]

c) (double weight) Use these boundary conditions to show that the scalar potential has the form

$$V(\boldsymbol{r}) = \begin{cases} -k_1 E_0 r\cos\theta, & r < a \\ -\left[1 - k_2\left(\frac{a}{r}\right)^3\right] E_0 r \cos\theta, & r > a \end{cases}, \tag{2}$$

where k_1 and k_2 are constants. Obtain expressions for the constants k_1 and k_2, and express your answer in terms of ε_1 and ε_2.

d) From the expression for the potential $V(\boldsymbol{r})$, Eq. (2), obtain the total electric field, $\boldsymbol{E}(\boldsymbol{r})$, inside and outside the sphere. Express one of the two terms for the electric field outside the sphere in terms of

$$\left[\frac{3\,(\boldsymbol{E}_0\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}}}{r^3} - \frac{\boldsymbol{E}_0}{r^3}\right], \tag{3}$$

where $\hat{\boldsymbol{r}}$ denotes a unit vector in the direction if $\boldsymbol{r}$.

e) What is the physical interpretation of the expression for the electric field (or potential) outside the sphere ($r > a$)? Obtain en expressions for the induced dipole moment of the dielectric sphere, $\boldsymbol{p}$, and its polarizability, α, defined via $\boldsymbol{p} = \alpha\boldsymbol{E}_0$. Determine the polarization of the sphere and show that it satisfies

$$\boldsymbol{P} \propto k_2\boldsymbol{E}_0. \tag{4}$$

f) Calculate the charge density, $\sigma(\theta)$, induced on the surface of the sphere. Make a sketch of $\sigma(\theta)$. Is the result as expected?

g) If the sphere is rotated at an angular frequency ω about the direction of the electric field $\boldsymbol{E}_0$, will a magnetic field be produced? Give reasons for your answer! (*Only* answering yes or no will give no credit.)

Problem 3.

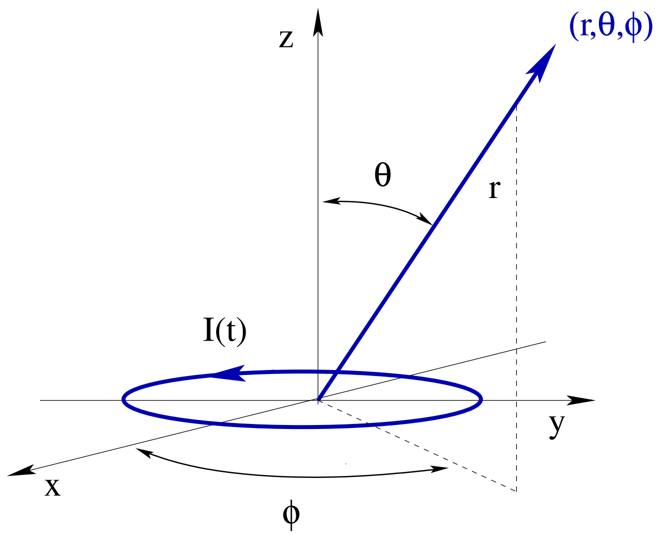

Figure 3: Schematics for problem 3

A small circular loop of wire has radius a and carries a time-dependent current

$$I(t) = I_0 \exp\left(-\mathrm{i}\omega t\right), \tag{5}$$

as shown in Fig. 3. Here ω denotes the angular frequency that is assumed to be a positive constant. It is also assumed that the wire-loop is placed in vacuum in the the xy-plane. In this problem, we aim to calculate the radiation pattern of the system.

To this end, we will start by obtaining some useful results that will be used later in the problem. In the Lorentz gauge, a general expression for the vector potential is

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3r' \,\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{6}$$

where $t_r = t - |\boldsymbol{r}-\boldsymbol{r}'|/c$ and the remaining quantities you should know the meaning of. For an *arbitrary* observation point, $\boldsymbol{r}$, the calculation of the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ is a non-trivial task. However, in the far field (the radiation zone), this calculation simplifies significantly. We recall that in the far field the following conditions apply $kr \gg 1$ (where $k = \omega/c$) and $r \gg r'$.

a) Assume a general time-harmonic current density of the form $\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r}|\omega)e^{-\mathrm{i}\omega t}$. Under the assumption of being in the *far field*, show that the vector potential (6) can be written in the form

$$\lim_{kr\to\infty}\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{e^{\mathrm{i}kr-\mathrm{i}\omega t}}{r}\boldsymbol{f}(\theta,\phi), \tag{7a}$$

where an angular factor is defined by

$$\boldsymbol{f}(\theta,\phi)=\int \mathrm{d}^3r'\ \boldsymbol{J}(\boldsymbol{r}'|\omega)e^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}. \tag{7b}$$

In passing we note that since one assumes $kr \gg kr' \gg 1$, the exponential contained in $\boldsymbol{f}(\theta,\phi)$, may be expanded and one may keep only the leading term.

b) Demonstrate that the mathematical expression for the current density for the geometry in Fig. 3 in spherical coordinates reads

$$\boldsymbol{J}(\boldsymbol{r},t)=\frac{I(t)}{a}\delta(r-a)\delta(\theta-\pi/2)\,\hat{\boldsymbol{\phi}}, \tag{8}$$

where $\hat{\boldsymbol{\phi}}$ is the unit vector in direction $\boldsymbol{\phi}$, and $\delta(\cdot)$ denotes a Dirac delta function. Explain in particular why a factor $1/a$ is needed in Eq. (8).

For the system in Fig. 3, the leading term of $\boldsymbol{f}(\theta,\phi)$ in the *far field* can be shown to be (like we did in the lectures)

$$\boldsymbol{f}(\theta,\phi)=-\mathrm{i}k\,\boldsymbol{m}_0\times\hat{\boldsymbol{r}}, \tag{9a}$$

with

$$\boldsymbol{m}_0=\frac{1}{2}\int \mathrm{d}^3r\,\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r}|\omega). \tag{9b}$$

c) Based on Eq. (9), show explicitly that

$$\boldsymbol{A}(\boldsymbol{r},t)\propto\sin\theta\,\frac{\exp\left(\mathrm{i}kr-\mathrm{i}\omega t\right)}{r}\,\hat{\boldsymbol{\phi}}. \tag{10}$$

What is the physical significance (meaning) of $\boldsymbol{m}(t)=\boldsymbol{m}_0e^{-\mathrm{i}\omega t}$?

d) (double weight) What is meant by radiation fields? Argue why it is sufficient to use Eq. (10) for the calculation of the *radiation* fields. Use this equation to calculate the electric and magnetic fields, $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$, respectively, in the *far field* [or radiation zone] ($kr \gg 1$).

The time averaged total power radiated from the system is defined as

$$P=\int \mathrm{d}\boldsymbol{A}\cdot\langle\boldsymbol{S}\rangle=\int \mathrm{d}\Omega\,\frac{\mathrm{d}P}{\mathrm{d}\Omega}, \tag{11}$$

where $\mathrm{d}\boldsymbol{A}$ is a small surface element (pointing outward), and

$$\langle\boldsymbol{S}\rangle=\frac{1}{2}\boldsymbol{E}\times\boldsymbol{H}^* \tag{12}$$

denotes the time-averaged Poynting vector. Moreover, $\mathrm{d}P/\mathrm{d}\Omega$ is the averaged power radiated per unit solid angle, also known as the differential power, and it defines the *radiation pattern* of the system.

e) Obtain an expression for $\langle \boldsymbol{S} \rangle$ in the *far field* and show that it satisfies

$$\langle \boldsymbol{S} \rangle \propto \frac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}}. \tag{13}$$

Use it to calculate the radiation pattern, $\mathrm{d}P/\mathrm{d}\Omega$. Make a sketch of the radiation pattern of the wire-loop. For what directions are the radiation the strongest and the weakest?

f) Calculate the total averaged power, P, radiated by the loop.

g) If the amplitude of the current in the wire is doubled, say, how much smaller must the radius of the loop be in order for the loop to radiate the same amount of power. Argue from the definition of the magnetic dipole moment, that you answer is reasonable.

Formulas

Some formulas that you may, or may not, need. The meaning of the symbols you should know.

$$\boldsymbol{p}(t) = \int \mathrm{d}^3 r\, \boldsymbol{r}\rho(\boldsymbol{r},t)$$

$$\boldsymbol{m}(t) = \frac{1}{2}\int \mathrm{d}^3 r\, \boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r},t)$$

$$\int_{-1}^{1}\mathrm{d}x\, P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{mn}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2}$$

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}P_n(\cos\theta')\left(\frac{r'}{r}\right)^n$$

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\, \mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\, \mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\, \mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\, \mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\, \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$