NTNU **Institutt for fysikk**

Contact during the exam:
Professor Ingve Simonsen
Telephone: 9 34 17 or 470 76 416

Exam in TFY4240 Electromagnetic Theory

Dec 03, 2014
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 6 pages.

This exam consists of three problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading. However, some sub-problems may be given double weight, but only if so is indicated explicitly.

For your information, it is estimated that you will spend about 35% of the available time on problem 1; 15% on problem 2; and about 50% on problem 3.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do a round 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Note that some formulas are given after the last problem!

Good luck to all of you!

Problem 1.

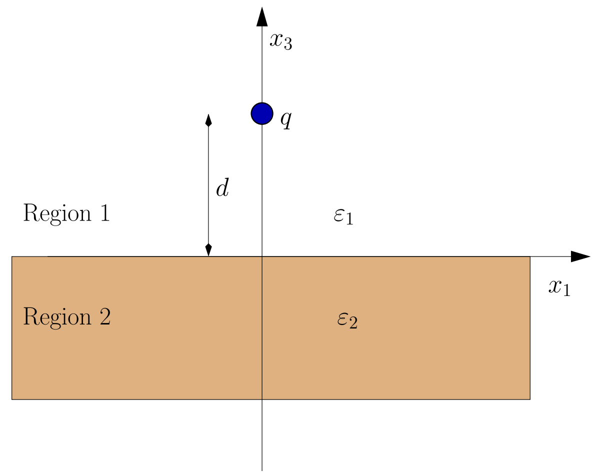

Figure 1: Schematics for problem 1

Consider the geometry depicted in Fig. 1 showing a charge q located a distance d above a flat interface ($x_3 = 0$). The medium where the charge is located is a dielectric and is characterized by a real and positive dielectric constant $\varepsilon_1 \geq 1$. Moreover, the medium filling the region below the surface has a dielectric constant ε_2 that can correspond to either a metal ($\operatorname{Re}\varepsilon_2 < 0$) *or* a dielectric ($\varepsilon_2$ is real and positive).

A coordinate system of position vector $\boldsymbol{r} = (x_1, x_2, x_3)$ is defined (see Fig. 1) so that its x_1x_2-plane coincides with the surface separating the two media, and the positive x_3 axis points upward. In this coordinate system, the interface is therefore defined by $x_3 = 0$ and the charge is located at $\hat{\mathbf{x}}_3 d$. The region $x_3 > 0$ will be refer to as region 1, and region 2 is where $x_3 < 0$.

In this problem, we are interested in solving the *electrostatic* problem for either a metallic or dielectric medium filling the region below the surface.

a) Which equations should the electric and displacment fields, $\boldsymbol{E}$ and $\boldsymbol{D}$, satisfy for the region (i) above the surface, and (ii) below the surface?

b) State (or derive) the equation that the scalar potential $V(\boldsymbol{r})$ should satisfy for the two regions. What boundary conditions should it satisfy at (i) $x_3 = 0$ and (ii) $r \to \pm\infty$.

First we will consider that the medium below the surface is a metal so that $\operatorname{Re}\varepsilon_2 < 0$. For simplicity, we will here assume that the metal is grounded[1].

c) Explain by your own words what the method of images is, and discuss where image charges are allowed to be placed. Determine the scalar potential $V(\boldsymbol{r})$ in region 1 by using the method of images. What is the potential in region 2?

In contrast to the previous subproblem, we will now assume that region 2 is filled by a dielectric and therefore characterized by a real and positive dielectric function ε_2. Now region 2 is *no longer* grounded.

[1] Strictly speaking, one does not here have to assume that the metal is grounded, as long as one neglects the effect of a thin surface (Stern) layer.

d) For this "dielectric case", we will again use the method of images. Write down *general* expressions for the potentials in region 1 and 2, denoted $V_+(\boldsymbol{r})$ and $V_-(\boldsymbol{r})$, respectively. In writing these expressions, you are expected to use a few unknown quantities (charges and distances). These quantities will be determined later.

e) (*Double weight*) Determine the unknown coefficients in $V_+(\boldsymbol{r})$ and $V_-(\boldsymbol{r})$ so that these potentials are expressed in terms of only known quantities (and the distance vector $\boldsymbol{r}$). [*Hint*: You will end up with a linear set of equations (2×2) that you need to solve.]

The expressions you (hopefully) derived in the previous subproblem for $V_+(\boldsymbol{r})$ are in general valid for any value of ε_2 (also for a metal).

f) Take the limit $|\varepsilon_2| \gg \varepsilon_1$ of the expressions derived for $V_+(\boldsymbol{r})$. Is the result reasonable?

Problem 2.

This problem is devoted to the conservation law for momentum. From the Maxwells equations with sources, written in terms of the fields $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, and $\boldsymbol{B}$, one may show that ($\partial_t \equiv \partial/\partial t$)

$$\partial_t \boldsymbol{g} + \boldsymbol{f} + \Big\{ \boldsymbol{D} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{E}\,(\boldsymbol{\nabla} \cdot \boldsymbol{D}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{H}) - \boldsymbol{H}\,(\boldsymbol{\nabla} \cdot \boldsymbol{B}) \Big\} = 0 \tag{1a}$$

where the electromagnetic momentum density is given by

$$\boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B}. \tag{1b}$$

All quantities appearing in Eq. (1) are assumed to depend on space and time "$(\boldsymbol{r}, t)$", but this dependence has not be indicated explicitly. For instance, this means that $\boldsymbol{f} \equiv \boldsymbol{f}(\boldsymbol{r}, t)$ and the same for all the electromagnetic fields.

Moreover, Eq. (1) can be written as

$$\partial_t \boldsymbol{g} + \boldsymbol{f} + \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{T}} = 0, \tag{2a}$$

where $\overleftrightarrow{\boldsymbol{T}}$ denotes Maxwell stress tensor that can be written as

$$\overleftrightarrow{\boldsymbol{T}} = T_{ij}\, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j, \tag{2b}$$

were T_{ij} represents its components, and $\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j$ denotes the tensor (or Kronecker) product between two of the units vectors $\hat{\mathbf{x}}_i$ ($i = 1, 2, 3$). This means that the curly brackets ($\{\cdot\}$) appearing in Eq. (1) can be written as $\boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{T}}$. [Recall that $\boldsymbol{a} \cdot (\boldsymbol{b} \otimes \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c}$.]

a) Derive Eq. (1a) (from the Maxwells equations) and identify an expression for $\boldsymbol{f}$ that appears in it. What is the physical interpretation of $\boldsymbol{f}$. Assume in your derivation (for simplicity) that we are in vacuum.

b) (*Double weight*) Show that the curly brackets ($\{\cdot\}$) in Eq. (1) indeed can be written as $\boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{T}}$, and identify the components T_{ij} of the tensor $\overleftrightarrow{\boldsymbol{T}}$ expressed in terms of the components of the fields $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, and $\boldsymbol{B}$. Demonstrate that Maxwell stress tensor is symmetric, *i.e.* $T_{ij} = T_{ji}$ ($i, j = 1, 2, 3$).

c) Explain by words (or math) what the physical meaning is of Eq. (2a).

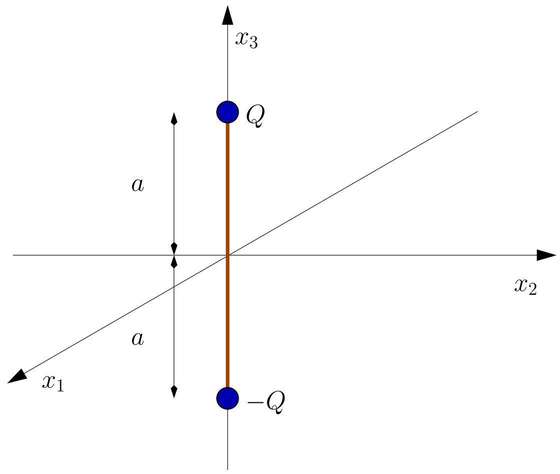

Figure 2: Schematics for problem 2

Problem 3.

Consider the situation shown in Fig. 2 where a perfect conducting thin wire of length $2a$ connects two (very) small charged metallic balls attached to its ends. The surrounding medium is assumed to be vacuum. Suppose the charge density of the system is time-harmonic and given by

$$\rho(\boldsymbol{r},t) = \rho(\boldsymbol{r}|\omega)\exp(-\mathrm{i}\omega t) = Q\left[\delta(x_3-a)-\delta(x_3+a)\right]\delta(x_1)\delta(x_2)\exp(-\mathrm{i}\omega t). \tag{3}$$

The current flows between the metallic balls through the thin wire[2]. In Eq. (3) the quantities a, Q and ω are constants and $\boldsymbol{r}=(x_1,x_2,x_3)$ denotes the position vector.

a) Calculate the time-dependent dipole moment $\boldsymbol{p}(t)$ for the system and obtain an expression for the current density $\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r}|\omega)\exp(-\mathrm{i}\omega t)$.

In the Lorentz gauge, the vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3r' \,\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{R}. \tag{4}$$

b) What is the meaning of the symbols $\boldsymbol{r}$, $\boldsymbol{r}'$, R and t_r in Eq. (4). Make a drawing indicating the quantities $\boldsymbol{r}$, $\boldsymbol{r}'$ and R. What is meant by the *radiation (or far) zone*? Which conditions should the quantities r, $k=\omega/c$ and a satisfy in this zone? Demonstrate that in this zone the vector potential can be written as (a multipole expansion)

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{\exp(\mathrm{i}kr-\mathrm{i}\omega t)}{r}\sum_{n=0}^{\infty}\frac{(-\mathrm{i}k)^n}{n!}\int \mathrm{d}^3r' \,\boldsymbol{J}(\boldsymbol{r}'|\omega)\left(\hat{\mathbf{r}}\cdot\boldsymbol{r}'\right)^n, \tag{5}$$

where $\hat{\mathbf{r}}$ denotes a unit vector in the direction of $\boldsymbol{r}$.

c) (*Double weight*) Explicitly calculate the lowest order (non-zero) contribution in expansion (5) and express your answer in terms of $\boldsymbol{p}(t)$. Use this result to find the corresponding electric field $\boldsymbol{E}(\boldsymbol{r},t)$, the magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$, and the time-averaged Poynting vector $\langle \boldsymbol{S}\rangle$. Also for this calculation we assume to be in the radiation zone.

[2]For this to happen, one has to drive the system somehow, but we will not be concerned with how this is done here.

d) Demonstrate that the radiation pattern, $\langle \mathrm{d}P/\mathrm{d}\Omega \rangle$, defined as the time-averaged power emitted per unit solid angle about direction (θ, ϕ), in general is given as

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \langle \boldsymbol{S} \rangle \cdot \hat{\mathbf{r}}\, r^2. \tag{6}$$

From this expression, calculate the radiation pattern within the same approximation used to obtain the results of the previous subproblem. Make a sketch of the radiation pattern and comment the results (do you recognize the pattern?).

We now aim at calculating the exact radiation pattern, *i.e.* without imposing the approximation used above.

e) Obtain the general expression for the vector potential valid at any distance r. Express your answer in terms of an integral over x_3'.

However, to be able to obtain analytic results, we will assume that the linear size of the system is small compared to the distance to the observer, *i.e.* we assume $r \gg a$.

f) Within this approximation ($r \gg a$) derive an expression for the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$. [Given answer :

$$\boldsymbol{A}(\boldsymbol{r}, t) = -\mathrm{i}\frac{\mu_0 c Q}{2\pi} \frac{\exp\left(\mathrm{i}kr - \mathrm{i}\omega t\right)}{r} \frac{\sin\left(ka\cos\theta\right)}{\cos\theta} \hat{\mathbf{x}}_3$$

g) (*Double weight*) For this exact case, calculate time-averaged Poynting vector $\langle \boldsymbol{S} \rangle$ (still $r \gg a$).

h) Based on the result for $\langle \boldsymbol{S} \rangle$, obtain an expression for $\langle \mathrm{d}P/\mathrm{d}\Omega \rangle$ now valid for all $r \gg a$. Make a sketch of the radiation pattern.

i) Take the $ka \ll 1$ limit of the expression for $\langle \mathrm{d}P/\mathrm{d}\Omega \rangle$ obtained in the previous subproblem and comment the result obtained. [Should you not be able to, or not have the time to, complete the previous subproblem, explain on physical grounds what you would expect in this limit.]

Formulas

Some formulas that you may, or may not, need. The meaning of the symbols you should know.

$$\boldsymbol{p}(t) = \int \mathrm{d}^3 r \, \boldsymbol{r} \rho(\boldsymbol{r}, t)$$

$$\boldsymbol{m}(t) = \frac{1}{2} \int \mathrm{d}^3 r \, \boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{r}, t)$$

$$\int_{-1}^{1} \mathrm{d}x \, P_m(x) P_n(x) = \frac{2}{2n+1} \delta_{mn}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\boldsymbol{p} \cdot \hat{\mathbf{r}}}{r^2}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \hat{\mathbf{r}}}{r^2}$$

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} P_n(\cos\theta') \left(\frac{r'}{r}\right)^n$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$		(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$		(permeability of free space)
$c = 3.00 \times 10^{8}\,\mathrm{m}/s$		(speed of light)
$e = 1.60 \times 10^{-19}\,\mathrm{C}$		(charge of the electron)
$m = 9.11 \times 10^{-31}\,\mathrm{kg}$		(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\left\{\begin{array}{lcl} x &=& r\sin\theta\cos\phi \\ y &=& r\sin\theta\sin\phi \\ z &=& r\cos\theta \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{x}} &=& \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} &=& \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} &=& \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{array}\right.$$

$$\left\{\begin{array}{lcl} r &=& \sqrt{x^2+y^2+z^2} \\ \theta &=& \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi &=& \tan^{-1}(y/x) \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{r}} &=& \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &=& \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &=& -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{array}\right.$$

Cylindrical

$$\left\{\begin{array}{lcl} x &=& s\cos\phi \\ y &=& s\sin\phi \\ z &=& z \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{x}} &=& \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} &=& \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} &=& \hat{\mathbf{z}} \end{array}\right.$$

$$\left\{\begin{array}{lcl} s &=& \sqrt{x^2+y^2} \\ \phi &=& \tan^{-1}(y/x) \\ z &=& z \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{s}} &=& \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} &=& -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} &=& \hat{\mathbf{z}} \end{array}\right.$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2ex] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2ex] \nabla \cdot \mathbf{B} = 0 \\[2ex] \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\[2ex] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2ex] \nabla \cdot \mathbf{B} = 0 \\[2ex] \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2ex] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2ex] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \qquad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \qquad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \qquad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \qquad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \qquad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \qquad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned}\textit{Curl}: \qquad \nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}\end{aligned}$$

$$\textit{Laplacian}: \qquad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \qquad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \qquad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \qquad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \qquad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$