NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Physics

Examination paper for TFY4240 Electromagnetic theory

Academic contact during examination: Associate Professor John Ove Fjærestad

Phone: 97 94 00 36

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Examination time (from-to): 9-13

Permitted examination support material: C

Approved calculator

Rottmann: Matematisk Formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of three problems, each containing several subproblems. The subproblems will be given approximately equal weight during grading. However, some subproblems may be given double weight, but only if this is indicated explicitly. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved. Some formulas can be found on the pages following the problems.

Language: English

Number of pages (including front page and attachments): 10

Checked by:

Date	Signature

Problem 1.

a) Briefly describe the "method of images" and the type of problems it can be used to solve.

A point charge q is held at a fixed position outside a grounded conducting sphere of radius R. The distance between the point charge and the center of the sphere is a. We take the z axis to pass through both the center of the sphere (where $z = 0$) and the point charge (where $z = a$). See Fig. 1 for an illustration of the geometry.

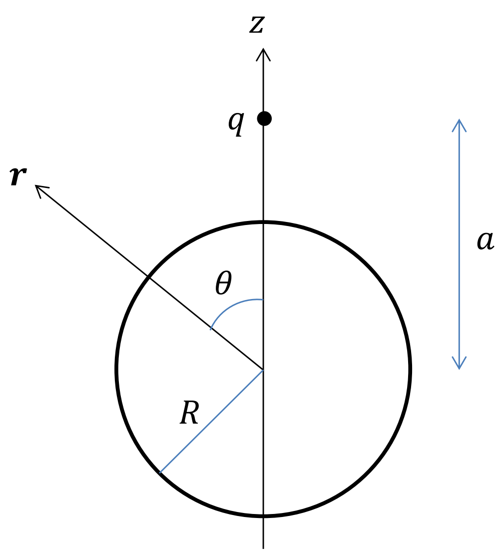

Figure 1: A point charge q outside a grounded conducting sphere of radius R.

b) (Double weight) We wish to find the potential V at an arbitrary point $\boldsymbol{r}$ outside the sphere. Show that the problem can be solved by introducing an image charge q' where

$$q' = -\frac{R}{a}q, \tag{1}$$

which is positioned on the z axis at $z = b$ where

$$b = \frac{R^2}{a}. \tag{2}$$

Give an expression for the potential $V(\boldsymbol{r})$ outside the sphere.

c) Find the induced surface charge density σ (which depends on the angle θ, see figure) and the total induced charge Q on the surface of the sphere.

d) Suppose next that the conducting sphere is not grounded, but is instead held at a fixed potential $V_0 \neq 0$ (with respect to infinity, where $V \to 0$). Again we wish to find the potential everywhere outside the sphere. Solve this problem by introducing one more image charge. What is the charge and position of this second image charge?

Problem 2.

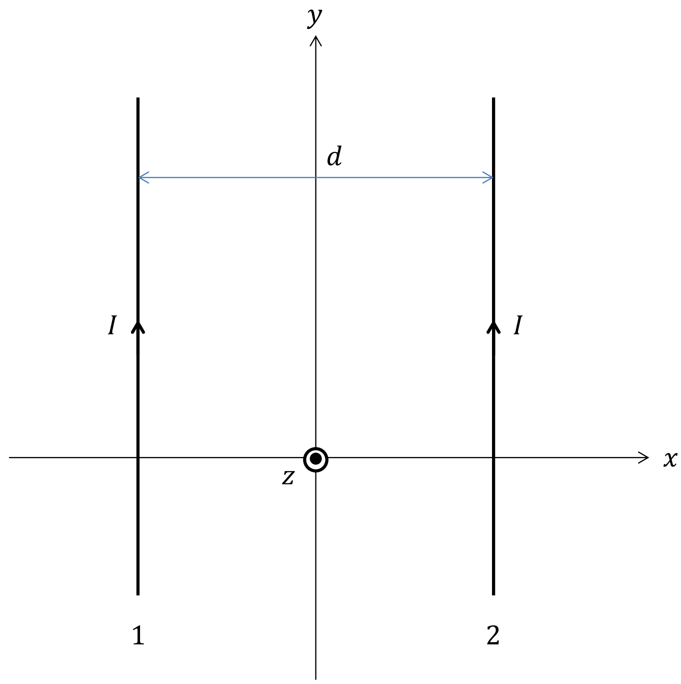

Figure 2: Two infinite straight wires a distance d apart.

Consider two infinitely long, straight, electrically neutral, parallel wires labeled 1 and 2 as shown in Fig. 2. The wires lie in the xy plane, are oriented along the y axis, and each wire carries the same current I in the positive y direction. The wires have x-coordinate $\pm d/2$, so the distance between the wires is d.

a) Show that the magnetic field produced by each wire has magnitude $\mu_0 I/2\pi r$ where r is the distance to the wire. Use this to find the force, per unit length, on wire 1. Is the force attractive or repulsive?

b) For a general problem in electrodynamics, briefly explain the meaning of the 3 terms in the equation

$$\boldsymbol{F} = \oint \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} - \frac{d}{dt}\int \frac{\boldsymbol{S}}{c^2}\, d^3r. \tag{3}$$

c) (Double weight) By an appropriate application of (3), give an alternative calculation of the force per unit length on wire 1. [Hint: In your calculation, give a key role to the plane $x = 0$ of points equidistant from both wires.]

Problem 3.

a) Show that in the Lorenz gauge, the electromagnetic potentials V and $\boldsymbol{A}$ satisfy the inhomogeneous wave equations

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0, \tag{4}$$

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}. \tag{5}$$

State the definition of the Lorenz gauge.

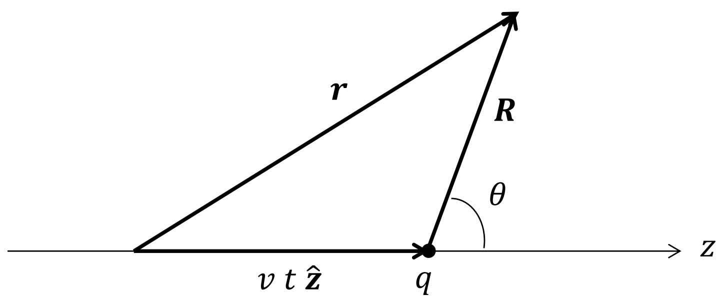

Figure 3: A particle of charge q moving with constant velocity $\boldsymbol{v}$ along the z axis.

In the following we will consider a derivation of the fields produced by a point charge q moving with **constant velocity**, by directly solving the inhomogeneous wave equations for the potentials for this special case. Taking the particle to be moving along the z axis with velocity $\boldsymbol{v} = v\hat{\boldsymbol{z}}$ with $v > 0$ (see Fig. 3), the source densities for the point charge are

$$\rho(\boldsymbol{r}, t) = q\delta(x)\delta(y)\delta(z - vt), \tag{6}$$

$$\boldsymbol{J}(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)\boldsymbol{v} \tag{7}$$

(we take the particle to be at the origin at $t = 0$). Since $\boldsymbol{J} \propto \hat{\boldsymbol{z}}$, the only nonzero component of $\boldsymbol{A}$ will be A_z. Furthermore, because of the uniform motion, the z and t dependence of the potentials must be through the combination $z - vt \equiv \xi$.

b) Show that in this case (4) can be rewritten as the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + (1-\beta^2)\frac{\partial^2 V}{\partial \xi^2} = -\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(\xi) \tag{8}$$

where $\beta = v/c$.

c) By making another change of variables from ξ to $z' = \gamma\xi$, where $\gamma = 1/\sqrt{1-\beta^2}$, rewrite (8) as a differential equation in the variables x, y, and z'. Based on the form of this differential equation, and using your knowledge of the solution of a mathematically related but physically simpler problem, show that V is given by

$$V(x, y, z, t) = \frac{\gamma q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}}. \tag{9}$$

d) Find the corresponding solution for $A_z(x, y, z, t)$. [Hint: Use the mathematical similarities between the problems for V and A_z.]

e) Show that the electric field is given by

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{\gamma q}{4\pi\epsilon_0} \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z - vt)\hat{\boldsymbol{z}}}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}. \tag{10}$$

f) Show that (10) can be rewritten as

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\boldsymbol{R}}}{R^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2\theta)^{3/2}}, \tag{11}$$

where $\boldsymbol{R}$ is the vector pointing from the position $vt\hat{\boldsymbol{z}}$ of the particle to the observation point $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, and θ is the angle between $\boldsymbol{R}$ and $\boldsymbol{v}$ (see Fig. 3).

g) Briefly discuss how the magnitude of $\boldsymbol{E}$ varies with direction θ for an ultrarelativistic particle ($\beta \approx 1$), especially comparing the forward/backward directions ($\theta \approx 0, \pi$) with the transverse directions ($\theta \approx \pi/2$). Do the same thing for a very nonrelativistic particle ($\beta \approx 0$).

h) Consider an (imagined) sphere of radius R centered on the particle at time t. What is the energy per unit time flowing through the surface of this sphere at time t?

i) Give a brief definition of **radiation** for a general problem in electrodynamics. For the special problem considered earlier, namely that of a particle moving with constant velocity, does the particle radiate? Justify your answer.

Formulas

Some formulas that you may or may not need (you should know the meaning of the symbols and possible limitations of validity):

$$\sigma = -\epsilon_0 \left[\frac{\partial V}{\partial n}\bigg|_{\text{outside}} - \frac{\partial V}{\partial n}\bigg|_{\text{inside}} \right] \tag{12}$$

$$\boldsymbol{F} = I\boldsymbol{\ell} \times \boldsymbol{B} \tag{13}$$

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \tag{14}$$

$$\delta(ax) = \frac{1}{|a|}\,\delta(x) \tag{15}$$

$$\nabla^2 V = -\rho/\epsilon_0 \tag{16}$$

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$