Problem 1.

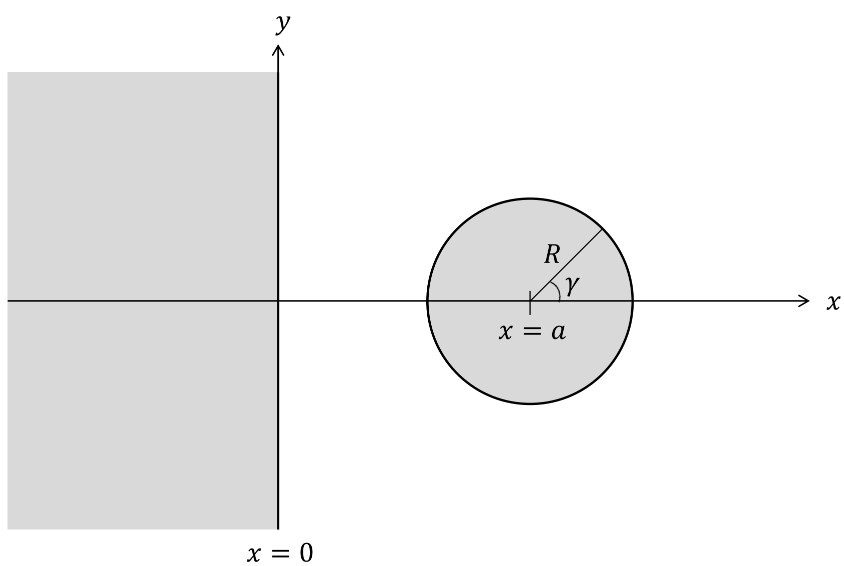
a) Find the electric field due to an infinite straight line with a static and uniform line charge density λ. Show that the scalar potential due to such a line that coincides with the z axis can be written as

$$V(x, y) = -\frac{\lambda}{4\pi\epsilon_0} \ln(x^2 + y^2) \tag{1}$$

(up to a constant).

Next, consider a different electrostatic problem involving two conductors (see the figure below). The right conductor is an infinitely long cylinder with radius R. The cylinder axis is parallel to the z axis and intersects the xy plane at $(a, 0)$. The left conductor fills the region $x \leq 0$, so that the infinite plane at $x = 0$ is the surface of this conductor.

Assume that the left conductor is grounded. Furthermore, the right conductor must also be at some uniform potential; call it V_{cyl} (we will return to its value later). We wish to find the potential V everywhere in the region outside the conductors (the white region in the figure).



This problem can be solved with the method of images by introducing *two* infinite straight image *lines* of charge, both lines being parallel to the z axis.

b) Use the boundary condition on the left conductor to find the *relative* positions and the *relative* line charge densities of the two image lines.

c) Use the boundary condition on the right conductor to find the positions and line charge densities of the two image lines. (The relation $(x - a)^2 + y^2 = R^2$ for the cylinder boundary may be helpful.)

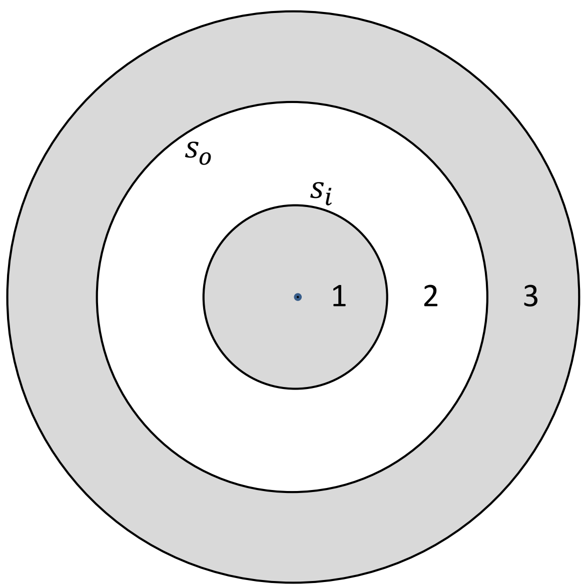
d) 1. The value of V_{cyl} is related to the line charge density Λ of the conducting cylinder (i.e. the total charge per unit length along the z direction). Determine this relation.

2. Assuming that $\Lambda > 0$, how do you expect that the surface charge density σ on the cylinder will vary (qualitatively, not quantitatively) with the angle γ shown in the figure?

(In order to answer questions 1 and 2, it is not necessary to calculate the surface charge density on the cylinder.)

Problem 2.

Consider a system whose cross section is shown in the figure below, and which extends infinitely in the direction perpendicular to the paper plane. Region 1 and region 3 are perfect conductors in which both $\boldsymbol{E}$ and $\boldsymbol{B}$ are zero. Region 2 is a simple nonconducting medium with electric permittivity ϵ and magnetic permeability μ, where both ϵ and μ are real and positive.



We introduce cylindrical coordinates $\boldsymbol{r} = (s, \phi, z)$ with the z axis going along the system center in the direction perpendicular to the paper plane. The interfaces between region 2 and the other regions are thus at $s = s_i$ (the inner (12) interface) and $s = s_o$ (the outer (23) interface).

Assume that in region 2 there is an electromagnetic wave of the form (using complex notation)

$$\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \tilde{E}_0(s) \exp[i(kz - \omega t)]\hat{\boldsymbol{s}} \qquad \text{and} \qquad \tilde{\boldsymbol{B}}(\boldsymbol{r}, t) = \tilde{B}_0(s) \exp[i(kz - \omega t)]\hat{\boldsymbol{\phi}} \tag{2}$$

with ω and k positive.

a) 1. At the two interfaces, consider those boundary conditions for the fields that do not involve charges or currents. Express them in terms of cylindrical components of the fields, and show that they are satisfied.

2. Find the s-dependence of $\tilde{E}_0(s)$ and $\tilde{B}_0(s)$, the ratio $\tilde{B}_0(s)/\tilde{E}_0(s)$, and the ratio ω/k.

b) Find the charge density ρ and current density $\boldsymbol{j}$ in region 2 (up to a complex constant prefactor).

c) For the special case of vacuum in region 2, i.e. $\epsilon = \epsilon_0$ and $\mu = \mu_0$, the equation

$$\frac{\partial \sigma}{\partial t} = -\nabla_{\text{2D}} \cdot \boldsymbol{K} \tag{3}$$

holds at the two interfaces. Here ∇_{2D} is the nabla operator in the 2-dimensional space defined by the interface (explicitly, $\nabla_{\text{2D}} = \hat{\boldsymbol{\phi}}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$ where $s = s_i$ or s_o). Briefly describe the meaning of the quantities σ and $\boldsymbol{K}$, what this equation expresses, and why this equation should hold.

d) Determine whether/how equation (3) is modified for the more general case, i.e. $\epsilon \neq \epsilon_0$ and $\mu \neq \mu_0$. Strengthen and check your reasoning with explicit calculations.

Problem 3.

A current is turned on at time $t = 0$. This current exists only in the xy plane, where it is everywhere the same. It flows in the x direction. The current crossing a line of unit length perpendicular to the current flow has a constant magnitude K. The xy plane is electrically neutral, and there is vacuum on both sides of it. We will be interested in the electromagnetic fields and various other quantities at an arbitrary point $\boldsymbol{r} = (x, y, z)$ outside the xy plane.

a) Show that the vector potential in the Lorenz gauge is[1]

$$\boldsymbol{A}(\boldsymbol{r}, t) = \begin{cases} 0 & \text{for } t < |z|/c, \\ \frac{\mu_0 K}{2}(ct - |z|)\,\hat{\boldsymbol{x}} & \text{for } t > |z|/c. \end{cases} \tag{4}$$

b) Find the electric and magnetic fields. Briefly comment on the results by comparing to a harmonic plane electromagnetic wave.

c) Find the energy density, energy current density, momentum density, and momentum current density.

d) Show that for any $(\boldsymbol{r}, t)$ such that $\boldsymbol{E}(\boldsymbol{r}, t) \neq 0$, only a specific finite region of the xy plane contributes to the electric field, and identify what this region is. [Hint: Calculate $\boldsymbol{E}$ in a different way than above, e.g. by using Jefimenko's equation.]

[1] The typo in the original version of the exam (the unit vector $\hat{\boldsymbol{x}}$ was missing) has been corrected.